INFIMUM OF THE METRIC ENTROPY OF HYPERBOLIC ATTRACTORS WITH RESPECT TO THE SRB MEASURE

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Abstract. Let M^n be a compact C^∞ Riemannian manifold of dimension $n \geq 2$. Let $\mathrm{Diff}^r(M^n)$ be the space of all C^r diffeomorphisms of M^n, where $1 < r \leq \infty$. For a C^r diffeomorphism f in $\mathrm{Diff}^r(M^n)$ with a hyperbolic attractor Λ_f on which f is topologically transitive, let $U(f)$ be the C^1 open set of $\mathrm{Diff}^r(M^n)$ such that each element in $U(f)$ can be connected to f by finitely many C^1 structural stability balls in $\mathrm{Diff}^r(M^n)$. Then by the structural stability, any element g in $U(f)$ has a hyperbolic attractor Λ_g and $g|\Lambda_g$ is topologically conjugate to $f|\Lambda_f$. Therefore, the topological entropy $h(g|\Lambda_g)$ is a constant function when it is restricted to $U(f)$. However, the metric entropy $h_\mu(g)$ with respect to the SRB measure $\mu = \mu_g$ can vary. We prove that the infimum of the metric entropy $h_\mu(g)$ on $U(f)$ is zero.

0. Introduction

Let M^n be a compact C^∞ Riemannian manifold of dimension $n \geq 2$, and $\mathrm{Diff}^r(M^n)$ be the space of all C^r-diffeomorphisms of M^n, $1 < r \leq \infty$. Suppose $f \in \mathrm{Diff}^r(M^n)$ has a hyperbolic attractor Λ_f. Consider an open set $U(f) \subset \mathrm{Diff}^r(M^n)$ in C^1-topology consisting of C^r diffeomorphisms g that has a hyperbolic attractor Λ_g topologically conjugate to Λ_f. (The precise definition of $U(f)$ is given in Section 1.) Since the hyperbolic attractors of any two elements in $U(f)$ are topologically conjugate, the topological entropy $h(g|\Lambda_g)$ is a constant function restricted to $U(f)$. On the other hand, every $g \in U(f)$ has an SRB measure μ_g on Λ_g. The metric entropy $h_{\mu_g}(g)$ can vary in $U(f)$. It has been shown that the dependence of μ_g on the map g is smooth when the maps involved have a higher degree of smoothness (see [Ru] and references therein). In this article, we prove that the infimum of the metric entropy $h_{\mu_g}(g)$ over $U(f)$ is zero.

In the special case of Anosov systems, if we denote by $\mathcal{A}^r(M^n)$ the set of all C^r Anosov diffeomorphisms of M^n, and let $U(f)$ be the connected component of $\mathcal{A}^r(M^n)$ in C^r topology that contains f, then by structure stability, all elements g in $U(f)$ are topologically conjugate and therefore the topological entropy $h(g)$ is constant over $U(f)$, but the metric entropy $h_{\mu_g}(g)$ over $U(f)$ with respect to the SRB measures μ_g can be arbitrarily small.

This result is interesting in several aspects. First, the topological entropy tells the global complexity of a dynamical system. If a dynamical system has a positive topological entropy, it can be thought as a chaotic dynamical system. However, the topological entropy can not tell the level of complexity of a chaotic dynamical system. As a macroscopic quantity, the metric entropy is the standard value that measures the level of complexity of a chaotic dynamical system. Our result says that, given a hyperbolic attractor, there is no barrier to reduce its metric entropy along a C^1 homotopic path to a number as small as one wishes, while preserving the uniform hyperbolicity and the topological entropy. Second, our construction of the homotopy gives an example where the connection between a global quantity such as metric entropy and local perturbations can be concretely described. Third, the result leads to many interesting questions one may ask about the nature of the variation of metric entropy within the open neighborhood $U(f)$. For example, is it true that the maximal value of the metric entropy $h_{\mu_g}(g)$ on $U(f)$ is the value of the topological entropy $h(f)$? Is there a way to perturb the diffeomorphism f in a direction so that its metric entropy $h_{\mu_g}(g)$ either decreases or increases monotonically? Are there any local extrema of the metric entropy $h_{\mu_g}(g)$ in $U(f)$?

To prove this result, the idea of perturbing a map at a fixed point in the direction of unstable manifold is quite natural. Since μ is an SRB measure, we have $h_\mu(f) = \int \log|\det Df_x|_{E^u_x}|d\mu$. So if we reduce the expanding rate near a fixed point p, a typical orbit will spend more time near p where $|\det Df_x|_{E^u_x}|$ is close to 1. But the approach by considering such an orbit is difficult because we do not know whether a typical orbit will remain typical after perturbation.

In order to get control of the SRB measure for perturbed maps, we need distortion estimates of the unstable Jacobian along orbits independent of the ever-decreasing expansion rate at the fixed point. We apply a technique used in studying almost Anosov systems ([HY], [Hu1]). However, in this paper, we deal with a family of diffeomorphisms $\{f_t : 0 \le t \le 1\}$ whose expansion decreases to 1 near a fixed point p and for the limiting map f_0, the expanding rate at p is 1. We obtain that the stable foliation is absolute continuous and the Jacobian of the holonomy maps are uniformly bounded for all f_t, and the distortion estimates of the unstable Jacobian are also uniformly bounded for all f_t as long as the initial points of the backward orbits are away from p.

Our construction also gives that every diffeomorphism with hyperbolic attractor is homotopic to a diffeomorphism that has an almost hyperbolic attractor, and has an infinite SRB measure on it.

The rest of the article is divided into 5 sections. The statement of results is given in the next section. In Section 2 we describe in detail how the perturbation is constructed. We show that any map $f \in \mathrm{Diff}^r(M^n)$ with a hyperbolic attractor can be perturbed successively in an appropriate way within $U(f)$ such that the perturbed map has a desired normal form in a neighborhood of its fixed point. In Section 4, we prove the uniform boundedness of the distortion of the unstable Jacobian for all perturbed maps. In Section 5, we give proofs of our theorems.

1. Statement of Results

For the definition of standard terms such as uniform hyperbolicity, topological conjugacy, Lyapunov exponents, topological and metric entropies, we refer readers to the book [KH].

Suppose that μ is an invariant measure for $f \in \mathrm{Diff}^r(M^n)$ that has positive Lyapunov exponents at almost every point x. Then f has a unstable manifold $W^u(x)$ at such x. A measurable partition ξ of M^n is said to be *subordinate to unstable manifolds* if for μ-a.e x, $\xi(x) \subset W^u(x)$ and $\xi(x)$ contains an open neighborhood of x in $W^u(x)$. Let $\{\mu_x^\xi\}$ denote a *canonical system of conditional measures* of μ with respect to ξ, that is, for every measurable set $B \subset M^n$, $x \to \mu_x^\xi(B)$ is measurable and

$$\nu(B) = \int_X \nu_x^\xi(B) d\nu(x).$$

(For a reference, see e.g. [Ro].) We say that a measure μ on Λ_f has *absolutely continuous conditional measures on unstable manifolds*, if for every measurable partition ξ subordinate to unstable manifolds, μ_x^ξ is absolutely continuous with respect to m_x^u for μ-a.e. $x \in \Lambda_f$, where m_x^u denotes the Lebesgue measure induced on $W^u(x)$ (see [LS] for more details). Now we give a definition for SRB measure which can be found in [BY] (see also e.g. [HY], [Hu1]).

Definition 1. *An f-invariant Borel probability measure μ on M is called an Sinai-Ruelle-Bowen measure or an SBR measure for $f : M \to M$ if*

i) *(f, μ) has positive Lyapunov exponents almost everywhere;*
ii) *μ has absolutely continuous conditional measures on unstable manifolds.*

It is well known that if μ is an SRB measure on a hyperbolic attractor of f, then for any continuous function ψ on M, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(f^i x) \to \int \psi d\mu$$

for Lebesgue almost every x in the basin of the attractor (see [S]). Also, an SRB measure μ is the only invariant measure satisfying the *variational principle*

$$h_\mu(f) = \int_\Lambda \log|\det Df_x|_{E_x^u}|d\mu, \tag{1.1}$$

where $h_\mu(f)$ is the metric entropy of f with respect to μ ([Bo]). This formula also follows from the entropy formula and the fact that the righthand side of (1.1) is equal to the integral of the sum of the positive Lyapunov exponents.

A map $f \in \mathrm{Diff}^r(M^n)$ is said to possess a hyperbolic attractor Λ_f if there is an open set $V \supset \Lambda_f$ such that f is hyperbolic on Λ_f and $\cap_{n=1}^\infty f^n(V) = \Lambda_f$.

Suppose $f \in \mathrm{Diff}^r(M^n)$ has a hyperbolic attractor Λ_f. By the structural stability, there is a sufficiently small ϵ-neighborhood of f with respect to C^1 topology (called a C^1 structural stability ball),

$$B_\epsilon^1(f) = \{g \in \mathrm{Diff}^r(M^n) \mid \|f-g\|_1 < \epsilon\},$$

where $\|\cdot\|_1$ means the C^1 norm, such that any $g \in B_\epsilon^1(f)$ has a hyperbolic attractor Λ_g and $g|\Lambda_g$ is topologically conjugate to $f|\Lambda_f$. We say that a map $g \in \mathrm{Diff}^r(M^n)$ can be *connected with* f if there are finitely many neighborhoods $\{B_{\epsilon_i}^1(f_i)\}_{i=0}^n$ such that $f_0 = g$ and $f_n = f$ and $f_i \in B_{\epsilon_{i+1}}^1(f_{i+1})$, $0 \le i \le n-1$. Let $U(f)$ be the collection of such diffeomorphisms g in $\mathrm{Diff}^r(M^n)$ which can be connected with f in the above sense. It is clear that $U(f)$ is an open set of $\mathrm{Diff}^r(M^n)$ with respect to the C^1-topology. If restricted to the hyperbolic attractors, any two maps in $U(f)$ are topologically conjugated by a Hölder continuous homeomorphism. Hence they have the same topological entropy.

For $g \in U(f)$, let μ_g be its SRB measure on Λ_g.

For convenience we assume that f is topologically transitive on Λ_f. Hence, all $g \in U(f)$ are topologically transitive on Λ_g, and the SRB measrues are unique. If otherwise we can use spectral decomposition ([Bo]) and consider a topologically transitive component instead.

Theorem A. *Suppose $f \in \mathrm{Diff}^r(M^n)$ has a hyperbolic attractor Λ_f on which f is topologically transitive. Then there is a C^1 path*

$$H = \{f_t \in U(f) \mid 0 < t \le 1\}$$

such that $f_1 = f$ and

$$\lim_{t \to 0^+} h_{\mu_t}(f_t) = 0, \tag{1.2}$$

where μ_t denotes the unique SRB measure of f_t on the hyperbolic attractor Λ_{f_t}.

Remark 1.1. *If f is an Anosov diffeomorphism, then $U(f)$ is a connected component of the subspace $\mathcal{A}^r(M^n)$ of all Anosov diffeomorphisms with respect* *to* C^1*-topology in* $\mathrm{Diff}^r(M^n)$*. The theorem says that on any connected component of* $\mathcal{A}^r(M^n)$*, the infimum of* $h_\mu(f)$ *is zero.*

Corollary 1.2. *As* $t \to 0$*,* $\mu_t \to \delta_{O(p)}$ *in the weak* topology, and*

$$h_\mu(f_t) \to 0 = h_{\delta_{O(p)}}(f_0),$$

where p *is a fixed or periodic point,* $\delta_{O(p)}$ *is the invariant measure supported on the orbit* $O(p)$ *of* p*.*

Our proof of the theorem also gives that the limiting diffeomorphism f_0 has an almost hyperbolic attractor Λ_{f_0}, on which f_0 admits an infinite SRB measure. A closed f-invariant subset $\Lambda \subset M$ is called an *almost hyperbolic set* if it is hyperbolic everywhere except for a finite set S. We refer [Hu2] for precise definition.

The following definition of an infinite SRB measure for an almost hyperbolic attractor can be found in [Hu1]. Here for a subset $\Gamma \subset M$, we denote by f_Γ the first return map on Γ, and by μ_Γ the normalization of $\mu|\Gamma$ as $\mu(\Gamma) < \infty$.

Definition 2. *An* f*-invariant Borel measure* μ *on* M *is called an infinite SBR measure, if* $\mu(M) = \infty$ *and for any open set* $V \supset S$*,*

i) $\mu(M\backslash V) < \infty$*,*
ii) $(f_{M\backslash V}, \mu_{M\backslash V})$ *has positive Lyapunov exponents almost everywhere, and* $\mu_{M\backslash V}$ *has absolutely continuous conditional measures on unstable manifolds of* f*.*

We say that maps f and g are C^r*-homotopy in a subset* $\mathcal{F} \subset \mathrm{Diff}^r(M^n)$ if there is a continuous map $H : [0,1] \times M^n \to M^n$ such that $H(0,\cdot) = f$, $H(1,\cdot) = g$, $H(t) = H(t,\cdot) \in \mathcal{F}\ \ \forall t \in [0,1]$, and $H(\cdot) : [0,1] \to \mathcal{F}$ is continuous with respect to C^r topology for $\mathcal{F}$. We also denote by $\overline{U(f)}$ the closure of $U(f)$ in C^r topology.

Theorem B. *Any diffeomorphism* f *in* $\mathrm{Diff}^r(M^n)$ *that has a hyperbolic attractor is* C^r*-homotopic to a diffeomorphism* f_0 *in* $\overline{U(f)}$ *that has an almost hyperbolic attractor on which* f_0 *admits an infinite SRB measure.*

Remark 1.3. *In the case of Anosov systems, the theorem implies that any Anosov diffeomorphism* f *is* C^r*-homotopic to an almost Anosov diffeomorphism in* $\overline{\mathcal{A}(M^n)}$ *that admits an infinite SRB measure.*

In [HY], it is proved that there is an almost Anosov diffeomorphism on the two dimensional torus that has an infinite SRB measure. This theorem generalizes the result to almost hyperbolic attractors and to higher dimensional manifolds.

Remark 1.4. *By the same arguments as in* [HY] *we can prove that* f_0 *does not admit SRB measures, and* $\lim_{n\to\infty}\sum_{i=0}^{n-1}\delta_{f^i x} = \delta_{O(p)}$ *for Lebesgue almost every point in the basin of the attractor.*

2. Construction

Suppose $f \in \mathrm{Diff}^r(M^n)$ has a hyperbolic attractor Λ_f on which f is topologically transitive. Without loss of generality, we will assume that f has a fixed point p. Otherwise, we can consider a periodic orbit and deform the maps in the same way near every point in the orbit. We will construct a family

$$\{f_t \in \overline{U(f)} \mid 0 \le t \le 1\}$$

of maps having hyperbolic attractors Λ_{f_t} on which f_t are topologically transitive. The family satisfies the following main properties:

(1) The fixed point p is preserved for the family and maps in the family are obtained by perturbing f in a small neighborhood of p.
(2) The contracting rates along the stable directions for the maps in the family are bounded above by a constant strictly less than 1, while the expanding rates along the unstable directions for the maps in the family in a small neighborhood of p can be arbitrarily close to 1 as $t \to 0^+$.

2.1. **Linearizing the map near p.** We first show the following lemma.

Lemma 2.1. *For the given map* f *as above, there is* $g \in \mathrm{Diff}^r(M^n)$ *having the same fixed point* p *such that the following properties hold:*

(1) $g \in U(f)$ *is* C^1 *close to* f*, and* C^r*-homotopy to* f *in* $U(f)$*.*
(2) g *is identical to* f *outside an* ϵ_1*−neighborhood of* p*.*
(3) *There is a coordinate system* $\eta : \mathbb{R}^n \to M^n$ *near* p *such that* $\eta(0) = p$ *and* $\eta^{-1}\circ g \circ \eta$ *is a linear map in an* ϵ_0 $(< \epsilon_1)-$ *open ball centered at* $0 \in \mathbb{R}^n$ *of the form*

$$L\begin{pmatrix} x^u \\ x^s \end{pmatrix} = \begin{pmatrix} Ax^u \\ Bx^s \end{pmatrix}, \tag{2.1}$$

where $x = (x^u, x^s)$ *are coordinates provided by the unstable and stable subspaces of* Df *at* p*,* A *is an expanding linear map, and* B *is an contracting linear map.*

Let n^u and n^s be the dimensions of the unstable and stable subspaces of Df at p. We may assume that the coordinate system provided by the unstable and stable subspaces are just Euclidean spaces $\mathbb{R}^{n^u}$ and $\mathbb{R}^{n^s}$ with $\mathbb{R}^{n^u} \otimes \mathbb{R}^{n^s} = \mathbb{R}^n$.

Proof. We identify p with the origin 0 of $\mathbb{R}^n$. There exists an ϵ_1-neighborhood of p in M^n such that f is C^1 close to its linear approximation $Df(0)$ in this ϵ_1-neighborhood. Identify this ϵ_1-neighborhood with an open ball $B_{\epsilon_1}(0)$ of radius ϵ_1 centered at 0 in $\mathbb{R}^n$.

In the coordinate system given by the unstable and stable subspaces $\mathbb{R}^{n^u}$ and $\mathbb{R}^{n^s}$, the linear operator $Df(0)$ takes the form

$$Df(0)\begin{pmatrix} x^u \\ x^s \end{pmatrix} = \begin{pmatrix} Ax^u \\ Bx^s \end{pmatrix},$$

Take a smaller open ball $B_{\epsilon_0}(0)$ of radius $0 < \epsilon_0 < \epsilon_1$ centered at 0. Extend the map $(f - Df(0))|_{B_{\epsilon_0}(0)}$ to a C^r map f_h on $B_{\epsilon_1}(0)$ so that $\|f_h\| + \|Df_h\|$ is small. This can be done as long as the ratio ϵ_0/ϵ_1 is small enough. Let

$$D(\epsilon_0) = \sup_{x\in B_{\epsilon_0}(0)} \{\|f(x) - Df(0)x\| + \|Df(x) - Df(0)x\|\}.$$

We can require that

$$\|f_h\| + \|Df_h\| \leq 2D(\epsilon_0).$$

Clearly,

$$\lim_{\epsilon_0\to 0} D(\epsilon_0) = 0.$$

Let $\tau(x)$ be a smooth map such that it is the identity inside $B_{\epsilon_0}(0)$ and it maps every point to zero outside of $B_{\epsilon_1}(0)$. Define

$$g = \begin{cases} f & \text{if } x \notin B_{\epsilon_1}(0); \\ f - \tau\circ f_h & \text{if } x \in B_{\epsilon_1}(0). \end{cases}$$

The map g is linear in $B_{\epsilon_0}(0)$. The C^1-distance between g and f is bounded by the $\|f_h\| + \|D\tau\|\|Df_h\|$. Note that by selecting a small ratio ϵ_0/ϵ_1, the map τ can be made as smooth as we wish with derivatives uniformly bounded. So, the C^1-distance between g and f can be is less than any given number. It implies that $g \in \mathrm{Diff}^r(M^n)$ is in a C^1-neighborhood of f. Thus, g has a hyperbolic attractor Λ_g on which g is topologically transitive. Clearly, g is C^r-homotopic to f in $U(f)$. This completes the proof. □

Note that, other than using $Df(0)x$, the linear approximation, we can use other nonlinear choices near 0. The only condition to be satisfied is that these choices are C^r and sufficiently C^1-close to f in an ϵ_1-neighborhood of 0.

Corollary 2.2. *The linear map in (2.1) near the origin* 0 *can be changed to any* C^r *map which is* C^1 *close to* f *in a neighborhood of* 0*, in particular, a* *map in the form of*

$$x = \begin{pmatrix} x^u \\ x^s \end{pmatrix} \to \begin{pmatrix} (A + C(x^s))x^u \\ Bx^s \end{pmatrix}, \tag{2.2}$$

where $C(x^s)$ are a linear maps for each x^s, depending on x^s in C^r and satisfying $C(0) = 0$.

Now we further perturb the map g inside the small neighborhood $B_{\epsilon_1}(0)$. The perturbation preserves the direct product structure form (2.1) near the point 0. We need to perturb the map along a homotopy path in $U(f)$ so that the matrices A and B become diagonal matrices first. In the unstable direction, we need to further reduce the expanding rates to a number close to 1. In the stable direction, the contracting rate will be a constant. The perturbation is no longer in a small C^1 neighborhood of f. But we will show that the perturbed maps are still uniformly hyperbolic and homotopic to f in $U(f)$.

Lemma 2.3. *Let g be the map obtained in Lemma 2.1. Assume that a C^r diffeomorphism $g_1 \in \mathrm{Diff}^r(M^n)$ has the following properties:*

1. *$g_1(x) = f(x)$ for $x \notin B_{\epsilon_0}(0)$ and $g_1(0) = 0$.*
2. *$g_1(x)$ preserves the direct product structure $\mathbb{R}^{n^u}$ and $\mathbb{R}^{n^s}$ for $x \in B_{\epsilon_0}(0)$, i.e., $g_1(x)$ takes the form*
$$g_1 \begin{pmatrix} x^u \\ x^s \end{pmatrix} = \begin{pmatrix} \tilde{A}(x^u) \\ \tilde{B}(x^s) \end{pmatrix} \quad \text{if } \ x = \begin{pmatrix} x^u \\ x^s \end{pmatrix} \in B_{\epsilon_0}(0),$$
where $\tilde{A}(x^u) \in \mathbb{R}^{n^u}$ and $\tilde{B}(x^s) \in \mathbb{R}^{n^s}$.
3. *$\tilde{A}(x^u)$ is expanding on $\mathbb{R}^{n^u}$ and $\tilde{B}(x^s)$ contracting on $\mathbb{R}^{n^s}$, both uniformly.*

Then, when ϵ_0 is sufficiently small, g_1 has a hyperbolic attractor Λ_g.

A direct corollary of this lemma is that when $\tilde{A}(x^u)$ and $\tilde{B}(x^s)$ are homotopic to the linear maps Ax^u and Bx^s in $\mathrm{Diff}^r(M^n)$, then g_1 is in $U(f)$.

Proof. Since we assume that f has a hyperbolic attractor Λ_f on which f is topologically transitive, so is g in Lemma 2.1. Let Λ_g be the corresponding hyperbolic attractor of g. Take an ϵ-neighborhood $O_\epsilon(\Lambda_g)$ such that the stable and unstable subspaces are extended to the entire neighborhood of the hyperbolic set Λ_g [KH]. We may also assume that $\epsilon_0 \le \epsilon$, thus the entire $B_{\epsilon_0}(0)$ is contained in $O_\epsilon(\Lambda_g)$. Take an appropriate coordinate system (almost Lyapunov metric) on the neighborhood $O_\epsilon(\Lambda_g)$ such that the stable and unstable subspaces are nearly orthogonal. At each point x inside the set $B_{\epsilon_0}(0)$, the unstable and stable subspaces E^u_x and E^s_x for $Dg(x)$ are within $C\epsilon_0$-distance

(Grassmannian distance) of $\mathbb{R}^{n^u}$ and $\mathbb{R}^{n^s}$, respectively, for some constant C. We assume that for some $\lambda < 1 < \mu$, we have

$$\|Dg(x)v\| \geq \mu\|v\|, v \in E^u_x; \quad \|Dg(x)w\| \leq \lambda\|w\|, w \in E^s_x.$$

To show that g_1 is uniformly hyperbolic, we first prove that both Dg_1 and $D^{-1}g_1$ admit an invariant cone field and they induce contracting operators on these cone fields. As a consequence, we obtain exponential splitting along any orbit of g_1 inside the open neighborhood $O_\epsilon(\Lambda_g)$. We then show the expanding and contracting rates along the invariant subspaces of the exponential splitting are uniform. In the process of the proof, we may need to further decrease the radius ϵ_0 for $B_{\epsilon_0}(0)$. But such modification will not affect the validity of the arguments since it depends only on the map g.

The cone field C^u_x along the unstable subspaces is defined as follows.

$$C^u_x = \begin{cases} \{(v,w) \in T_xM^n : v \in E^u_x, w \in E^s_x, \|w\| \leq \alpha\|v\| \text{ for } x \notin B_{\epsilon_0}(0)\}; \\ \{(v,w) \in T_xM^n : v \in \mathbb{R}^{n^u}, w \in \mathbb{R}^{n^s}, \|w\| \leq \alpha\|v\| \text{ for } x \in B_{\epsilon_0}(0)\}, \end{cases}$$

where $0 < \alpha < 1$. The cone field C^s_x along the stable subspaces is defined in a similar way. Note that this cone field is not continuous. But this should not affect the Hölder continuity of stable and unstable subspaces because of invariance.

We just need to show that the derivative operator is contracting on this cone field, since invariance follows the contraction. Given a point x, if both x and $g_1(x)$ are inside or outside of $B_{\epsilon_0}(0)$, the contraction follows automatically since E^u and E^s are invariant under Dg_1 when it is restricted to the outside of $B_{\epsilon_0}(0)$ and since $\mathbb{R}^{n^u}$ and $\mathbb{R}^{n^s}$ are invariant under Dg_1 when it is restricted to the inside of $B_{\epsilon_0}(0)$. So, we just need to verify two situations: $x \in B_{\epsilon_0}(0)$ but $g_1(x) \notin B_{\epsilon_0}(0)$; and $g_1(x) \in B_{\epsilon_0}(0)$ but $x \notin B_{\epsilon_0}(0)$.

Now we consider the case $x \in B_{\epsilon_0}(0)$ but $g_1(x) \notin B_{\epsilon_0}(0)$.

Take $(v,w) \in C^u_x, v \in E^u_x, w \in E^s_x$. Then $Dg_1(x)(v,w) = (D\tilde{A}v, D\tilde{B}w)$ is in $\mathbb{R}^{n^u} \oplus \mathbb{R}^{n^s}$ coordinate system. In the $E^u \oplus E^s$ coordinate system, we have

$$(P_{11}D\tilde{A}v + P_{12}D\tilde{B}w, P_{21}D\tilde{A}v + P_{22}D\tilde{B}w) \in E^u \oplus E^s,$$

where P_{ij} denotes the coordinate change matrices and $D\tilde{A}, D\tilde{B}$ are derivative operators of $\tilde{A}(x^u)$ and $\tilde{B}(x^s)$. Since p is a fixed point of g and g_1 is C^0-close to g, $\|g_1(x)\| \leq 2\epsilon_0$. So, we have $\|P_{11}\|, \|P_{22}\| \geq (1-\epsilon)$ and $\|P_{12}\|, \|P_{21}\| \leq \epsilon$ where $\epsilon = C\epsilon_0$ for some constant C. Estimating the norms, we have

$$\|P_{11}D\tilde{A}v + P_{12}D\tilde{B}w\| \geq ((1-\epsilon)\mu - \alpha\epsilon\tilde{\lambda})\|v\|;$$

$$\|P_{21}D\tilde{A}v + P_{22}D\tilde{B}w\| \leq (\epsilon\tilde{\mu} + \alpha\lambda(1-\epsilon))\|v\|,$$

where $\tilde{\mu}, \tilde{\lambda}$ denote the maximal expanding rate and minimal contracting rate of $\tilde{A}$ and $\tilde{B}$, respectively. We have

$$\|P_{21}D\tilde{A}v + P_{22}D\tilde{B}w\| \leq \alpha \cdot \frac{\lambda(1-\epsilon)+\epsilon\tilde{\mu}/\alpha}{(1-\epsilon)\mu - \alpha\epsilon\tilde{\lambda}} \cdot \|P_{11}D\tilde{A}v + P_{12}D\tilde{B}w\|. \tag{2.3}$$

If ϵ_0 is sufficiently small, we have

$$\frac{\lambda(1-\epsilon)+\epsilon\tilde{\mu}/\alpha}{(1-\epsilon)\mu - \alpha\epsilon\tilde{\lambda}} < 1.$$

The proof for the other case is completely parallel. Thus we proved invariance of the cone field around the stable subspaces.

We now prove that g_1 is uniform hyperbolic. We only consider the unstable direction. The proof for stable direction is the same. Since p is a fixed point, we can take $0 < \epsilon_2 \leq \epsilon_0$ such that $B_{\epsilon_2}(0) \cup g_1 B_{\epsilon_2}(0) \cup g_1^{-1} B_{\epsilon_2}(0) \subset B_{\epsilon_o}(0)$. So for any x, either both x and $g_1(x)$ are outside of $B_{\epsilon_2}(0)$, or both x and $g_1(x)$ are inside of $B_{\epsilon_0}(0)$. Note that the estimates obtained in (2.3) remain valid if we replace ϵ_0 by a smaller number ϵ_2. Thus, if both x and $g_1(x)$ are outside of $B_{\epsilon_2}(0)$, taking $(v,w) \in C_x^u, \|w\| = \tilde{\alpha}\|v\|$ for some $0 \leq \tilde{\alpha} \leq \alpha < 1$, and using the invariant splitting of g, we have

$$\begin{aligned} & \|Dg_1(x)(v,w)\| = \|(Dg_1(x)v, Dg_1(q)w)\| \\ = \ & \max(\|Dg_1(x)v\|, \|Dg_1(x)w\|) = \|Dg_1(x)v\| \geq \mu\|v\|, \end{aligned}$$

where we use an equivalent Finsler metric in estimating the expanding rates. If both x and $g_1(x)$ are inside of $B_{\epsilon_0}(0)$, using the $\mathbb{R}^{n^u}$ and $\mathbb{R}^{n^s}$ coordinates, we have the same estimates in Finsler metric. We thus conclude that g_1 is uniformly hyperbolic. ☐

Corollary 2.2 and Lemma 2.3 give us plenty of freedom to perturb the map f inside a fixed neighborhood $B_{\epsilon_0}(0)$. In order to obtain bounded distortion along unstable manifold, we will need to use Lemmas 2.1 and 2.3 to obtain a map $g_1 \in U(f)$ in the form of $(x^u, x^s) \to (\tilde{A}(x^u), \tilde{B}(x^s))$. Then, we apply Corollary 2.2 to obtain another map in $U(f)$ that has the form $(x^u, x^s) \to (\tilde{A}(x^u) + C(x^s)x^u, \tilde{B}(x^s))$ near the fixed point p.

2.2. **Constructing the map f_t.**

We now construct a family

$$H = \{f_t \in U(f) \mid 0 \leq t \leq 1\}$$

such that the infimum of metric entropies of maps in this family with respect to their SRB measures is zero.

Recall that n^s and n^u are dimensions of $\mathbb{R}^{n^s}$ and $\mathbb{R}^{n^u}$, respectively. Take an even integer m such that $n^u < m \leq n^u + 2$, and denote $\beta = 1/m$. Hence,

$\beta n^u < 1$. Take a family of smooth increasing functions $\phi_t : [0, 1/2) \to \mathbb{R}^+$ that satisfies the following conditions.

(1) *$\phi_t(r) = (1 - r^m)^{-\beta}$ for $r \geq t$,*
(2) *$\phi_t(r) = (1 - (t/2)^m)^{-\beta} > 1$ is a constant for $0 \leq r \leq t/2$,*
(3) *$\phi_t''(r)$ is bounded uniformly in t and nonnegative when $\frac{t}{2} \leq r \leq t$.*

Note that if $r = 1/n^\beta \in [t, 1/2)$, then $\phi'(r) > 0$ and

$$\frac{1}{n^\beta}\phi_t(\frac{1}{n^\beta}) = \frac{1}{(n-1)^\beta}. \tag{2.4}$$

When $t = 0$, $r\phi_0(r)$ is essentially a special Manneville-Pomeau map.

Take two open neighborhoods $\Omega_0 \subset \Omega_1$ such that $p \in \Omega_0 \subset \overline{\Omega}_0 \subset \Omega_1$. Denote $\kappa^s = \|Df(0)|_{E_0^s}\|$, the contracting rate at 0 in the stable direction. Define a family of diffeomorphisms f_t such that

(1) *for $x \in \Omega_0$,*

$$f_t \begin{pmatrix} x^u \\ x^s \end{pmatrix} = \begin{pmatrix} [\phi_t(|x^u|) + |x^s|^2]x^u \\ \kappa^s x^s \end{pmatrix}, \quad \textit{if } \ x = \begin{pmatrix} x^u \\ x^s \end{pmatrix}; \tag{2.5}$$

(2) *for $x \notin \Omega_1$, $f_t(x) = f(x)$;*
(3) *f_t is uniformly hyperbolic for all $t > 0$ and the contracting rate is independent of t;*
(4) *the expanding rate outside of Ω_0 is bounded below by a constant $\mu > 1$ with μ independent of t.*

We also require that the second derivative of f_t is uniformly bounded for all t and $x \in M^n$. For simplicity, we use $|\cdot|$ to denote both the absolute value of a number and the Euclidean norm of a vector. The existence of f_t is guaranteed by Lemma 2.1, Corollary 2.2, and Lemma 2.3 when t is sufficiently small. We note that f_t is C^∞ on Ω_0 since m is an even number. It is clear that f_t is C^r-homotopic to f for all $0 < t \leq 1$. Hence, f_t has an hyperbolic attractor Λ_{f_t}, and therefore $f_t \in U(f)$ for all $0 < t \leq 1$ by the statement after Lemma 2.3.

3. Preliminaries

Being uniformly hyperbolic, the maps constructed in the previous section have many special properties. We list some of them with only sketches of proofs since details can be found in many books such as [KH, HPS]. Let $E_x^u(f_t)$ and $E_x^s(f_t)$ be the stable and unstable subspaces with respect to f_t.

Lemma 3.1. *The maps $x \to \{E_x^u(f_t)\}$ and $x \to \{E_x^s(f_t)\}$ are Hölder continuous and the Hölder exponents and constants can be chosen in a way independent* *of* $0 \le t \le 1$. *More precisely, there exist* $\gamma > 0$, $H > 0$ *such that for all* $t > 0$, $x, x' \in M^n$,

$$d\big(E^u_x(f_t), E^u_{x'}(f_t)\big),\ d\big(E^s_x(f_t), E^s_{x'}(f_t)\big) \le H d(x, x')^\gamma,$$

where the distance between subspaces is the Grassmannian distance.

Proof. Note that $\|Df_t(x)|_{E^u_x(f_t)}\| \ge 1$ and $\|Df_t(x)|_{E^s_x(f_t)}\| \le \lambda^s < 1$ for all $x \in \Lambda_{f_t}$ and $t \ge 0$. The conclusions follow from the fact that the Hölder exponent and constant depend only on the Lipschitz constant of the map f_t and the gap between the expansion and contraction rates. See [HPS]. □

For $\varepsilon > 0$, we denote

$$E^u_{x,f_t}(\varepsilon) = \{v \in E^u_x(f_t) :\ |v| \le \varepsilon\} \quad \text{and} \quad E^s_{x,f_t}(\varepsilon) = \{v \in E^s_x(f_t) :\ |v| \le \varepsilon\}$$

and

$$E_{x,f_t}(\varepsilon) = E^u_{x,f_t}(\varepsilon) \times E^s_{x,f_t}(\varepsilon).$$

Proposition 3.2. *For each* $t \ge 0$, *there exist two continuous foliations* $\mathcal{F}^u(f_t)$ *and* $\mathcal{F}^s(f_t)$ *on* Λ_{f_t} *tangent to* $E^u(f_t)$ *and* $E^s(f_t)$ *respectively for which the following statements hold.*

(1) *The leaf of* $\mathcal{F}^s(f_t)$ *through* x, *denoted by* $\mathcal{F}^s(x, f_t)$, *is the stable manifold at* x, *i.e.*

$$\begin{aligned}&\mathcal{F}^s(x, f_t) = W^s(x, f_t)\\ &= \{x' \in \Lambda_{f_t} :\ \exists C = C'_x,\ s.t.\ d(f^n_t(x), f^n_t(x')) \le C(\kappa^s_t)^n\ \forall n \ge 0\},\end{aligned}$$

where κ^s_t *denotes the contracting rate of* f_t *on the stable manifold.*

(2) *The leaf of* $\mathcal{F}^u(f_t)$ *through* x, *denoted by* $\mathcal{F}^u(x, f_t)$, *is the unstable or weak unstable manifold at* x, *i.e.*

$$\mathcal{F}^u(x, f_t) = W^s(x, f_t) = \{x' \in \Lambda_{f_t} :\ \lim_{n\to\infty} d(f^{-n}_t(x), f^{-n}_t(x')) = 0\}.$$

(3) *There exist constants* $\delta > 0$ *and* $D > 0$ *such that for all* $t \ge 0$, $x \in \Lambda_{f_t}$, *if* $\mathcal{F}^u_\delta(x, f_t)$ *is the component of* $\mathcal{F}^u(x, f_t) \cap \exp_x E_{x,f_t}(\delta)$ *containing* x, *then* $\exp^{-1}_x \mathcal{F}^u_\delta(x, f_t)$ *is the graph of a function*

$$\phi^u_{x,f_t} :\ E^u_{x,f_t}(\delta) \to E^s_{x,f_t}(\delta)$$

with $\phi^u_{x,f_t}(0) = 0$ *and* $\|\phi^u_{x,f_t}\|_{C^r} \le D$. *The analogous statement holds for* $\mathcal{F}^s_\delta(x, f_t)$.

Proof. There results follow from Theorem 5.5 and Theorem 5A.1 in [HPS]. □

For convenience we will write $W^u(x, f_t) = \mathcal{F}^u(x, f_t)$, $W^u_\delta(x, f_t) = \mathcal{F}^u_\delta(x, f_t)$, etc. and refer to $W^u(x, f_t)$ and $W^u_\delta(x, f_t)$ as the "unstable manifold" and "local unstable manifold" respectively at x.

Further, for simplicity, we use E^u, E^s, W^u and W^s to stand for invariant subspaces and sub-manifolds, when doing so will not cause much confusion.

For $x' \in W^s(x, f_t)$, let $d^s(x, x')$ denote the distance between x and x' measured along $W^s(x, f_t)$ with the induced metric. For $x'' \in W^u(x, f_t)$, the distance $d^u(x, x'')$ is defined similarly.

Lemma 3.3. *There is $J^s > 0$ such that for any $x \in \Lambda_{f_t}$, $x' \in W^s(x, f_t)$,*

$$\frac{|\det Df_t^n(x)|_{E_x^u(f_t)}|}{|\det Df_t^n(x')|_{E_{x'}^u(f_t)}|} \le J^s\big(d^s(x,x')\big)^\gamma, \quad \forall t \ge 0.$$

Proof. It follows from the standard arguments and the fact that f_t is uniformly contracting along the stable manifold and $E_x^u(f_t)$ is Hölder continuous. $\square$

One of the important ingredients in the proof of the main theorem is that the W^s-foliation for f_t is absolute continuous with a uniformly bounded Jacobian.

Let Δ_1 and Δ_2 be two W^u leaves for f_t. A holonomy map $\theta : \Delta_1 \to \Delta_2$ is defined by sliding along the W^s leaves for f_t, i.e. for $x \in \Delta_1$, $\theta(x) \in \Delta_2 \cap W^s(x, f_t)$. We have the following proposition.

Proposition 3.4. *Given $D_1 > 0$, there exists $J_1^s > 0$ such that for every $(\Delta_1, \Delta_2; \theta)$ with $d^s(x, \theta(x)) < D_1$ $\forall x \in \Delta_1$, for every $x' \in \Delta_1$ and $\varepsilon > 0$ with $B^u(x', \varepsilon) \subset \Delta_1$,*

$$m^u(B^u(x',\varepsilon)) \le J_1^s m^u\big(\theta B^u(x',\varepsilon)\big).$$

Proof. Let $\mathcal{D}$ be any small disk in Δ_1. We will argue that $m^u(\mathcal{D}) \approx m^u(\theta\mathcal{D})$, where m^u denotes the Lebesgue measure on W^u-sub-manifolds for f_t, and "$\approx$" means "up to a constant".

Let

$$\kappa_-^s = \kappa_-^s(f_t) = \max\{\|Df_t^{-1}(x)|_{E_x^s(f_t)}\|^{-1} : \ x \in \Lambda_{f_t}\},$$

that is, κ_-^s is the minimal norm of Df_t restricted to the stable bundle $E^s(f_t)$.

By taking a sufficiently large iterate of f_t, we may assume that $f_t^n(\mathcal{D})$ and $f_t^n\big(\theta\mathcal{D}\big)$ are close sufficiently so that the "diameters" of $f_t^n(\mathcal{D})$ and $f_t^n(\theta\mathcal{D})$ are much larger than the distance $d^s(y, \theta y)$ for any $y \in f_t^n(\mathcal{D})$. Here we also use θ to denote the holonomy map from $f_t^n(\Delta_1)$ to $f_t^n(\Delta_2)$.

Take a finite cover $\{B^u(y_i, r)\}_{i=1}^k$ of $f_t^n\mathcal{D}$ consisting balls of radius r, where $r = 3(\kappa_-^s)^n$, such that for any $y' \in f_t^n(\mathcal{D})$, there are at most C_1 such balls covering this point, where C_1 only depends on the dimension of W^u.

Since $d^s(y', \theta y') \le (\kappa_-^s)^n$ for any $y' \in f_t^n(\mathcal{D})$, we know that $\theta B^u(y_i, r)$ contains a ball of radius at least $(\kappa_-^s)^n$ and is contained in a ball of radius at most $5(\kappa_-^s)^n$. Therefore, we have

$$m^u(B^u(y_i,r)) \approx m^u(\theta B^u(y_i,r)). \tag{3.1}$$

By Lemma 3.3, we have

$$\left|\det Df_t^n(x')|_{E^u_{x'}(f_t)}\right| \approx \left|\det Df_t^n(\theta x')|_{E^u_{\theta x'}(f_t)}\right| \quad \forall x' \in f_t^{-n}B^u(y_i,r). \tag{3.2}$$

Using the fact that the C^2 norm of f_t is uniformly bounded, $W^u(x,f_t)$ are C^r leaves, and f_t is expanding on $W^u(x,f_t)$, we get for all $x_1', x_2' \in f_t^{-n}B^u(y_i,r)$,

$$\frac{\left|\det Df_t^n(x_1')|_{E^u_{x_1'}(f_t)}\right|}{\left|\det Df_t^n(x_2')|_{E^u_{x_2'}(f_t)}\right|} \approx \prod_{j=0}^{n-1}\bigl(1 \pm \mathrm{const}\cdot d^u(f_t^j x_1', f_t^j x_2')\bigr) \tag{3.3}$$

$$\le\bigl(1 \pm \mathrm{const}\cdot \operatorname{diam} B^u(y_i,r)\bigr)^n \approx \bigl(1 \pm \mathrm{const}\cdot D_1(3\kappa_-^s)^n\bigr)^n \approx \mathrm{const}.$$

Combining (3.1)-(3.3), we get

$$m^u\bigl(f_t^{-n}B^u(y_i,r)\bigr) \approx m^u\bigl(f_t^{-n}(\theta B^u(y_i,r))\bigr). \tag{3.4}$$

Let S be the union of $f_t^{-n}B^u(y_i,r)$ that belong to $\mathcal{D}$. Since

$$\operatorname{diam}(f_t^{-n}B^u(y_i,r)),\ \operatorname{diam}(\theta\bigl(f_t^{-n}B^u(y_i,r)\bigr)) \to 0 \quad \text{as} \quad n\to\infty,$$

it follows that

$$m^u(S) \approx m^u(\mathcal{D}) \quad \text{and} \quad m^u(\theta S) \approx m^u(\theta\mathcal{D}).$$

Also recall that each point in $\mathcal{D}$ belongs to at most C_1 sets of the form $f_t^{-n}B^u(y_i,r)$, then (3.4) implies $m^u(\mathcal{D}) \approx m^u(\theta\mathcal{D})$. □

4. Distortion Estimates

We now estimate the distortion of the Jacobian along the unstable direction. We use the $\mathbb{R}^{n^u} \oplus \mathbb{R}^{n^s}$-coordinate near the fixed point 0 and use Euclidean metric for convenience. Let

$$S = S_\delta = \{x = (x^u, x^s) \in \Omega_1 : |x^u|, |x^s| \le \delta\}.$$

Denote $S^+ = \overline{f_t(S) \setminus S}$. It is clear that S^+ is independent of t for all small t.

Let $\delta^+ = \delta\phi_t(\delta)$, where ϕ_t is given in Subsection 2.2. It is the outer radius of S^+.

We denote $y = (y^u, y^s) = (y_0^u, y_0^s)$ and $z = (z^u, z^s) = (z_0^u, z_0^s)$. We also denote $f_t^{-i}y = y_i = (y_i^u, y_i^s)$ and $f_t^{-i}z = z_i = (z_i^u, z_i^s)$.

Lemma 4.1. *Let $\delta > 0$ be sufficiently small. For any $x \in S^+ \setminus W^u_{\delta^+}(0,f_t)$, if*

$$d(x, W^u_{\delta^+}(0,f_t)) \to 0,$$

then

$$d(f_t^{-n}(x), W^s_\delta(0,f_t)) \to 0,$$

and the convergence is uniform for all $x \in S^+$ and $t \in I$, where $n = n(x)$ is the largest integer such that $f_t^{-1}x, \cdots, f_t^{-n}x \in S$.

Proof. First, we note that for any $z \in S^+ \cap W^u_{\delta+}(0, f_t)$, $f_t^{-j} z \to 0$ uniformly with z and t as $j \to \infty$, since this is true for $t = 0$ and $f_t^{-j} z$ converges to 0 faster than $f_0^{-j} z$ does.

To consider the case $x \in S^+ \setminus W^u_{\delta+}(0, f_t)$, we claim that for any $\hat{y}$ and $\hat{z}$, $|\hat{y}^s| > |\hat{z}^s|$ and $|\hat{y}^u| \le |\hat{z}^u|$ imply $|\hat{y}^u_1| \le |\hat{z}^u_1|$. Otherwise, there would be a point $\hat{x}$ in a curve joining $\hat{y}$ and $\hat{z}$ near which $|(f_t^{-1}\hat{x})^u|$ increases with $|\hat{x}^s|$. It contradicts the fact that $|(f_t^{-1}x)^u|$ decreases with $|x^s|$ for $x = (x^u, x^s)$.

This claim implies inductively that for $y = x = (x^u, x^s)$ and $z = (x^u, 0)$, we always have $|y^u_i| \le |z^u_i|$ for all $0 \le i \le n$. So if $x \in S^+ \setminus W^u_{\delta+}(0, f_t)$, we have $d(x_n, W^s_\delta(0, f_t)) \le d(z_n, W^s_\delta(0, f_t))$. Note that f_t^{-1} is expanding at a constant rate in the stable direction, we get $n = n(x) \to \infty$ uniformly as $d(x, W^u_{\delta+}(0, f_t)) \to 0$. Hence the result of the lemma follows. □

Lemma 4.2. *Given any $\delta > 0$, there exist $J_0 > 0$ independent of t such that for any $x \in S^+$, $y, z \in W^u(x, f_t) \cap S^+$,*

$$\log \frac{\left|\det Df_t^{-n}(z)|_{E^u_z(f_t)}\right|}{\left|\det Df_t^{-n}(y)|_{E^u_y(f_t)}\right|} \le J_0 d(y, z), \tag{4.1}$$

whenever $f_t^{-1}x, \cdots, f_t^{-n}x \in S$.

Proof. We assume that δ is small enough such that $S^+ \subset \Omega_0$. So we can use (2.5) for the map f_t.

We only need to consider the case when t is small since f_t is uniformly hyperbolic for all t away from 0.

Further, we only need to consider the case when x is sufficiently close to $W^u_{\delta+}(0, f_t)$, since otherwise, the time that the backward orbit of x leaves the set S is bounded, and the inequality (4.1) surely holds. Further, we may assume that n is the largest integer such that $f_t^{-1}x, \cdots, f_t^{-n}x \in S$.

Since the angles between $E^u_{y_i}(f_t)$, $E^u_{z_i}(f_t)$ and $\mathbb{R}^{n^u}$ are exponentially decreasing as i change from n to 0, we just need to show

$$\log \frac{\left|\det Df_t^{-n}(z)|_{\mathbb{R}^{n^u}(f_t)}\right|}{\left|\det Df_t^{-n}(y)|_{\mathbb{R}^{n^u}(f_t)}\right|} \le J_0 d(y, z). \tag{4.2}$$

By (2.5), we have

$$Df_t(x)|_{\mathbb{R}^{n^u}} = \big(\phi_t(|x^u|) + |x^s|^2\big) I + x^u \cdot \phi_t'(|x^u|)|x^u|^{-1}(x^u)^T,$$

where I is the $n^u \times n^u$ identity matrix, and $(x^u)^T$ is the transpose of the column vector x^u. Using the fact that $\det(aI + bxx^T) = a^{n-1}(a + b|x|^2)$ for an $n^u \times n^u$ matrix $aI + bxx^T$, we have

$$|\det Df_t(x)|_{\mathbb{R}^{n^u}}| = \big(\phi_t(|x^u|) + |x^s|^2\big)^{n^u - 1}\big(\phi_t(|x^u|) + |x^s|^2 + \phi_t'(|x^u|)|x^u|\big). \tag{4.3}$$

It means that $|\det Df_t(x)|_{\mathbb{R}^{n^u}}|$ only depends on the norms $|x^u|$ and $|x^s|$.

Case One: y^u, z^u and $p^u = 0^u$ are on the same line: $z^u = ay^u$ for some $0 < a < 1$.

Note that $|z_n^s|$ is close to δ by the choice of n. So $|z_n^s| - |z_{n-1}^s|$ is close to $(1-\kappa^s)\delta$. By the previous lemma we know that if $d(x, W_{\delta^+}^u(0, f_t))$ is sufficiently small, then all x_n, y_n, z_n are close to $W_{\delta^+}^s(0, f_t)$ such that

$$||y_n^s| - |z_n^s|| \le d(y_n, z_n) < |z_n^s| - |z_{n-1}^s|,$$

and the amount $d(x, W_{\delta^+}^u(0, f_t))$ can be taken to be independent of x and t. Then we get $|y_n^s| > |z_{n-1}^s|$. By (2.5), it implies $|y_i^s| > |z_{i-1}^s|$ for all $1 \le i \le n$ and in particular, $|y_1^s| > |z^s|$.

Since $y_1 \in S$, we have $|y_1^u| \le \delta$ by the definition of S. Also, since $z \in S^+$, we have $|z^u| > \delta \ge |y_1^u|$. Therefore, by the fact $|y_i^s| > |z_{i-1}^s|$ and $|y_1^u| \le |z^u|$, we can use the claim of the proof in the previous lemma with $\hat{y} = y_i$ and $\hat{z} = z_{i-1}$ to get inductively that $|y_i^u| \le |z_{i-1}^u|$ for all $1 \le i \le n$. It follows that

$$\sum_{i=0}^{j}(|y_i^u| - |z_i^u|) = |y_0^u| - (|z_0^u| - |y_1^u|) - \cdots - (|z_{j-1}^u| - |y_j^u|) - |z_j^u| < |y_0^u| \le \delta^+$$

for any $0 \le j \le n$.

Now we refine the estimates. Let

$$\begin{aligned}\tau &= \{\hat{x} = (\hat{x}^u, \hat{x}^s) \in W_\delta^u(x, f_t) : \hat{x}^u = by^u, a \le b \le 1\},\\ \bar{\tau} &= \{\bar{x} = (\bar{x}^u, \bar{x}^s) \in W_\delta^u(x, f_t) : \bar{x}^u = by^u, b > 0, \delta \le |\bar{x}^u| \le \delta^+\},\end{aligned}$$

and $\tau_j = f_t^{-j}\tau$. Both τ and $\bar{\tau}$ are curves in the unstable manifold $W_{\delta^+}^u(x, f_t)$ whose u-component are on the line segment from z^u to y^u and from $\delta y^u/|y^u|$ to $\delta^+ y^u/|y^u|$ respectively. We denote by $\bar{y}$ and $\bar{z}$ the endpoints of the curve $\bar{\tau}$.

Let $\pi^u : \Omega_1 \to \mathbb{R}^{n^u}$ be the projection, and denote $\tau^u = \pi^u\tau$ and $\tau_j^u = \pi^u\tau_j$. We know that the first component of the map f_t^j sends τ_j^u to τ^u. Also recall that z^u and y^u are in the same direction. By (2.5) we know that all z_j^u and y_j^u are in the same direction as well. Hence τ_j^u is a segment of a straight line. Let $E \subset \mathbb{R}^{n^u}$ be the one dimensional subspace containing $(y^u, 0^s)$ and $(z^u, 0^s)$, and let ℓ denote the arc length. We get

$$\big||y_j^u| - |z_j^u|\big| = \ell(\tau_j^u) = \int_{\tau^u} \big\|Df_t^{-j}|_E\big\| d\ell. \tag{4.4}$$

Also note that Df_t preserves the subspace $E \oplus \mathbb{R}^{n^s}$. Hence $Df_t^j(x)|_E = Df_t(f_t^{j-1}x)|_E \cdots Df_t(x)|_E$. By the fact that $\big\|Df_t^{-1}|_E\big\|$ is uniformly bounded away from 0, and the C^2 norm of f_t is uniformly bounded from above on Ω_0, we have that for $j \le n$,

$$\log \frac{\|Df_t^{-j}(z)|_E\|}{\|Df_t^{-j}(y)|_E\|} = \log \prod_{i=0}^{j-1} \Big(1 + \frac{\|Df_t^{-1}(z_i)|_E\| - \|Df_t^{-1}(y_i)|_E\|}{\|Df_t^{-1}(y_i)|_E\|}\Big) \tag{4.5}$$

$$\le C \sum_{i=0}^{j-1} \Big| \|Df_t^{-1}(z_i)|_E\| - \|Df_t^{-1}(y_i)|_E\| \Big| \le CD \sum_{i=0}^{j-1} \big| |y_i^u| - |z_i^u| \big| \le I_0,$$

where we take $I_0 = \delta^+ CD$. Since this is true for any $y, z \in \bar{\tau}$, by (4.4) we obtain

$$\frac{\big||y_j^u| - |z_j^u|\big|}{\ell(\tau^u)} \le e^{I_0} \frac{\big||\bar{y}_j^u| - |\bar{z}_j^u|\big|}{\ell(\bar{\tau}^u)},\ j = 0, 1, \cdots, n.$$

Note that $\ell(\bar{\tau}^u) = \delta^+ - \delta$ only depends on δ, $\ell(\tau^u) = d(y^u, z^u)$, and $\sum_{i=0}^{j} (|\bar{y}_i^u| - |\bar{z}_i^u|) \le \delta^+$. We get

$$\sum_{i=0}^{j} \big||y_j^u| - |z_j^u|\big| \le \frac{e^{I_0}\delta^+}{\delta^+ - \delta} d(y^u, z^u) \le I_1 d(y^u, z^u)$$

for some $I_1 = I_1(\delta)$ independent of t.

Now repeating the arguments as for (4.5), we get

$$\log \frac{\big|\det Df_t^{-j}(z)|_{\mathbb{R}^{n^u}}\big|}{\big|\det Df_t^{-j}(y)|_{\mathbb{R}^{n^u}}\big|} \le CD \sum_{i=0}^{j-1} d(y_i, z_i) \le J' d(y^u, z^u), \tag{4.6}$$

where $J' = J'(\delta)$ is independent of t.

Case Two: $|y^u| = |z^u|$.

We may assume that $|y^s| \le |z^s|$. By (4.3), we know that $|\det Df_t(x)^{-1}|_{\mathbb{R}^{n^u}}|$ is at least quadratic in terms of $|x^s|$. Thus, we have that there is $c_1 > 0$ of order $|z^s|$ such that if $|y^u| = |z^u|$, then

$$\big|\det Df_t(z)^{-1}|_{\mathbb{R}^{n^u}} - \det Df_y(y)^{-1}|_{\mathbb{R}^{n^u}}\big| \le c_1\big(|z^s| - |y^s|\big),$$

and therefore

$$\log \frac{\big|\det Df_t(z)^{-1}|_{\mathbb{R}^{n^u}}\big|}{\big|\det Df_t(y)^{-1}|_{\mathbb{R}^{n^u}}\big|} \le c_1 C\big(|z^s| - |y^s|\big) \le C_2\big(|z^s| - |y^s|\big). \tag{4.7}$$

Also, by (2.5), there is a constant $c_0 > 0$ of order $|y^u||z^s|$ such that

$$\big||z_1^u| - |y_1^u|\big| \le c_0\big(|z^s| - |y^s|\big).$$

Take $\tilde{z}_1 = (\tilde{z}_1^u, \tilde{z}_1^s) \in W_\delta^u(x_1, f_t)$ such that $\tilde{z}_1^u = (|y_1^u|/|z_1^u|) z_1^u$. That is, $\tilde{z}_1$ is the point on $W^u(x_1, f_t)$ whose u-coordinate $\tilde{z}_1^u$ is proportional to z_1^u and satisfies

$|\tilde z_1^u| = |y_1^u|$. Then we denote $\tilde y_1 = y_1$. Hence, by (4.6) we have

$$\log \frac{\big|\det Df_t^{-n-1}(z_1)|_{\mathbb{R}^{n^u}}\big|}{\big|\det Df_t^{-n-1}(\tilde z_1)|_{\mathbb{R}^{n^u}}\big|} \le J'\big(|z_1^u| - |y_1^u|\big) \le C_3\big(|z^s| - |y^s|\big),$$

where $C_3 = c_0 J'$. By (4.7) and the fact $\tilde y_1 = y_1$, this inequality implies

$$\log \frac{\big|\det Df_t^{-n}(z)|_{\mathbb{R}^{n^u}}\big|}{\big|\det Df_t^{-n}(y)|_{\mathbb{R}^{n^u}}\big|} \le \log \frac{\big|\det Df_t^{-n-1}(\tilde z_1)|_{\mathbb{R}^{n^u}}\big|}{\big|\det Df_t^{-n-1}(\tilde y_1)|_{\mathbb{R}^{n^u}}\big|} + (C_2 + C_3)\big(|z^s| - |y^s|\big).$$

We denote $\tilde y_0^s = y^s$, $\tilde z_0^s = z^s$ and choose $\tilde z_i$ and $\tilde y_i$ in a similar way as we choose $\tilde z_1$ for $i = 2, 3, \cdots, n-1$, depending on whichever is larger between $|\tilde z_{i-1}^s|$ and $|\tilde y_{i-1}^s|$. Inductively, we have

$$\log \frac{\big|\det Df_t^{-n}(z)|_{\mathbb{R}^{n^u}}\big|}{\big|\det Df_t^{-n}(y)|_{\mathbb{R}^{n^u}}\big|} \le (C_2 + C_3)\sum_{i=0}^{n-1}\big||\tilde z_i^s| - |\tilde y_i^s|\big|. \tag{4.8}$$

We observe that all $f_t^{-n+i}\tilde z_i$ and $f_t^{-n+i}\tilde y_i$ are in $W_\delta^u(x_{n-1}, f_t)$ for $0 \le i \le n-1$. It is clear that

$$d((f_t^{-n+i}\tilde y_i)^u, (f_t^{-n+i}\tilde z_i)^u) \le d(y, z).$$

So there is $C_4 > 0$ such that

$$d((f^{-n+i}\tilde y_i)^s, (f^{-n+i}\tilde z_i)^s) \le C_4 d(y, z), \quad 0 \le i \le n-1$$

because $d((f^{-n+i}\tilde y_i)^s, (f^{-n+i}\tilde z_i)^s)$ is dominated by $d((f^{-n+i}\tilde y_i)^u, (f^{-n+i}\tilde z_i)^u)$ on unstable manifolds close to $W_\delta^u(p, f_t)$. Since f_t is contracting in $\mathbb{R}^{n^s}$ direction with the rate κ_t^s, we get that

$$d(\tilde y_i^s, \tilde z_i^s) \le C_4(\kappa_t^s)^{n-i} d(y, z).$$

Hence by (4.8), we get

$$\log \frac{\big|\det Df_t^{-n}(z)|_{\mathbb{R}^{n^u}}\big|}{\big|\det Df_t^{-n}(y)|_{\mathbb{R}^{n^u}}\big|} \le J'' d(y, z). \tag{4.9}$$

Case Three: The general case, $y, z \in W^u(x, S^+)$.

We take $\tilde y = (\tilde y^u, \tilde y^s) \in W^u(x, S^+)$ such that $y^u, \tilde y^u$ and 0^u are on the same line and $|\tilde y^u| = |z^u|$. Then from Case One and Two we get (4.6) and (4.9) with y, z replaced by $y, \tilde y$ and $\tilde y, z$ respectively. Therefore we obtain (4.2) with $J_0 = J' + J''$. □

Proposition 4.3. *Given any $\delta > 0$, there exist constants $\delta' > 0$ and $J > 1$ such that for all $t \in [0, 1]$, $x \in \Lambda_{f_t}$, $y, z \in W^u(x, f_t) \setminus S_\delta$ with $d^u(y, z) \le \delta'$,*

$$J^{-1} \le \frac{\big|\det Df_t^{-n}(z)|_{E_z^u(f_t)}\big|}{\big|\det Df_t^{-n}(y)|_{E_y^u(f_t)}\big|} \le J \qquad \forall n \ge 0. \tag{4.10}$$

Proof. Using Lemma 4.2 and the fact that f_t is uniformly hyperbolic outside $P_\delta(f_t)$, we can get the result by the same arguments identical to that in the proof of Proposition 3.1 in [HY]. $\square$

The inequalities (4.10) immediately lead to the following estimate of the density function of SRB measure on unstable manifolds.

Corollary 4.4. *Given any $\delta > 0$, there exist constants $\delta' > 0$ and $J > 1$ such that for any $t \in [0,1]$, SRB measure $\mu = \mu_t$ of f_t, and partition $\xi = \xi_t$ subordinate to unstable manifolds of f_t, the density functions $\rho_{x,t}$ of the conditional measures μ_x^ξ of μ with respect to Lebesgue measure m_x^u satisfies*

$$J^{-1} \le \frac{\rho_{x,t}(z)}{\rho_{x,t}(y)} \le J$$

for μ-a.e. $x \in \Lambda_{f_t}$, and for all $y, z \in \xi(x) \setminus S_\delta$ with $d^u(y,z) < \delta'$.

Proof. It is well known (see e.g. [L]) that the density function $\rho_{x,t}$ satisfies

$$\frac{\rho_{x,t}(y)}{\rho_{x,t}(z)} = \frac{\prod_{i=0}^{\infty} \left|\det Df_t^{-1}(z_i)|_{E^u_{z_i}(f_t)}\right|}{\prod_{i=0}^{\infty} \left|\det Df_t^{-1}(y_i)|_{E^u_{y_i}(f_t)}\right|}$$

for all $y, z \in W^u(x, f_t)$. Then we use Proposition 4.3. $\square$

We mention here that we can always increase δ' by increasing J.

5. Proofs of Theorems A and B

Consider a general map f that has a hyperbolic invariant set Λ.

The map f has local product structure, that is, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $y, z \in \Lambda$ with $d(y,z) \le \delta$, $[y,z] = W^s_\varepsilon(y) \cap W^u_\varepsilon(y)$ contains exact one point.

A rectangle R in Λ is a set such that $y, z \in R$ implies $[y,z], [z,y] \in R$. If $\mathcal{D}^u$ and $\mathcal{D}^s$ are pieces of W^u and W^s leaves for f respectively, then $[\mathcal{D}^u, \mathcal{D}^s]$ denotes the rectangle $\{[y,z] : \ y \in \mathcal{D}^u, z \in \mathcal{D}^s\}$ whenever everything makes sense. If R is a rectangle and $x \in R$, we let

$$W^u(x,R) = W^u(x,R,f) = W^u_\delta(x,f) \cap R,$$
$$W^s(x,R) = W^s(x,R,f) = W^s_\delta(x,f) \cap R.$$

If Q and R are two rectangles, we say that $f^n(Q)$ *u-crosses* R if $\forall x \in Q$ with $f^n x \in R$, $f^n W^u(x,Q) \cap R = W^u(f^n x, R)$.

A *Markov partition* of Λ for f is a set of finite number of rectangles $\{R_i\}$ on Λ with $\overline{\operatorname{int} R_i} = R_i$, such that (a) $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ whenever $i \neq j$, and (b) fR_i u-crosses R_j and $f^{-1}R_j$ s-crosses R_i whenever $fR_i \cap R_j \neq \emptyset$.

Now we assume that f is as in Theorem A that has a hyperbolic attractor Λ_f.

Take a Markov partition $\mathcal{R} = \{R_i\}$ of Λ_f. We assume that the fixed point p near which we deformed the map is contained in the interior $\operatorname{int} R_0$ of an element R_0 of the Markov partition, and that Ω_1 is small enough such that $\Omega_1 \subset \operatorname{int} R_0$. Hence, the perturbation is inside $\operatorname{int} R_0$. If there is no fixed point contained in the interior of any element of the Markov partition, we can choose a periodic orbit such that every point in the orbit is contained in the interior of some R_i of the Markov partition, and make the same perturbation near every point in the orbit.

Denote $\partial^s R_i = \{x \in R_i : x \notin \operatorname{int} W^u(x, R_i)\}$, and $\partial^s \mathcal{R} = \cup_i \partial^s R_i$. The properties of Markov partition imply that $f(\partial^s \mathcal{R}) \subset \partial^s \mathcal{R}$. Since $\partial^s \mathcal{R}$ consists of stable manifolds of f, and is not perturbed when we construct f_t, it also consists of stable manifolds of f_t for all $t \in [0, 1]$. Similarly, $\partial^u \mathcal{R}$ consists of unstable manifolds of f_t for all t. Hence, $\mathcal{R}$ is a Markov partition for all f_t.

Take $\delta > 0$ small enough such that $B^u(p, \delta) \subset W^u(p, R_0)$. Let

$$P_t = [B^u(p, \delta), W^s(p, R_0)],$$

the rectangle determined by $B^u(p, \delta)$ and $W^s(p, R_0)$. Clearly, $P_t \subset R_0$, and $W^s(x, R_0, f_t) = W^s(x, P_t, f_t)$ if $x \in P_t$, and $W^s(x, R_0, f_t) \cap P_t = \emptyset$ otherwise.

Denote

$$Q_t = f_t^{-1} P_t \setminus P_t.$$

Since f_t has a hyperbolic attractor Λ_{f_t} on which f_t is topologically transitive, f_t has a unique SRB measure μ_t on Λ_{f_t} for all $0 < t \leq 1$. Let

$$\nu_t = \frac{1}{\mu_t(\Lambda_{f_t} \setminus P_t)} \mu_t. \tag{5.1}$$

We have $\nu_t(\Lambda_{f_t} \setminus P_t) = 1$.

Lemma 5.1. *There is $c > 0$ such that $\nu_t(Q_t) > c$ for all $0 < t \leq 1$.*

Proof. Suppose there exists $\{t_n\} \in (0, 1]$ such that $\nu_{t_n}(Q_{t_n}) \to 0$.

Since each f_t is topologically transitive on Λ_{f_t}, for each rectangle R_i, $i > 0$, there is $k = k(i) \geq 0$ independent of t such that $f_t^k R_i \cap Q_t \neq \emptyset$. Note that if $f_t^{k'} R_i \cap R_0 \neq \emptyset$ for some $k' > 0$, then $f_t^{k'} R_i$ u-cross R_0 by the properties of Markov partition, and therefore $f_t^{k'-j} R_i$ u-crosses Q_t for some $1 \leq j \leq k'$. So we may assume that $k(i)$ is chosen in such a way that $f_t^l R_i \cap R_0 = \emptyset$ for all $l = 1, \cdots, k$.

Since the hyperbolicity of f_t is uniform outside R_0 for all $0 < t \leq 1$, we know that there is $c_1 = c_1(i) > 0$ independent of t such that

$$m^u(W^u(x, f_t^{-k}Q_t, f_t)) \geq c_1 \cdot m^u(W^u(x, R_i, f_t))$$

for all $x \in f_t^{-k}Q_t \cap R_i$.

Denote by $\xi = \xi_t$ the partition of Λ_{f_t} into $\{W^u(x, R_i, f_t) : x \in \Lambda_{f_t}, R_i \in \mathcal{R}\}$. So ξ_t is a partition subordinate to unstable manifolds of f_t. Denote by $\nu^\xi_{x,t}$ the corresponding conditional measure of μ_t on $\xi(x)$, where $\xi(x)$ is the element of ξ containing x. Note that ν_t and μ_t have the same conditional measure on $\xi(x)$. Since μ_t is an SRB measure, we can denote by $\rho_{x,t}$ the density function of $\nu^\xi_{x,t}$ with respect to the Lebesgue measure m^u_x on $W^u(x, R_i, f_t)$. We know that by Corollary 4.4 the ratio $\rho_{x,t}(y)/\rho_{x,t}(z)$ is bounded away from 0 and infinity for any $y, z \in W^u(x, R_i, f_t)$, and the bounds can be chosen in a way that is independent of $0 < t \leq 1$ and x. So we know that there is $c_2 > 0$ such that

$$\nu^\xi_{x,t}(W^u(x, f^{-k}Q_t, f_t)) \geq c_2 \cdot \nu^\xi_{x,t}(W^u(x, R_i, f_t)).$$

Consequently, by invariance of ν_t, we get

$$\nu_t(Q_t) \geq \nu_t(f_t^{-k}Q_t \cap R_i) \geq c_2 \cdot \nu_t(R_i).$$

It follows

$$\nu_{t_n}(Q_{t_n}) \geq c_2 \cdot \nu_{t_n}(R_i).$$

Note that $P_t \subset R_0$. Suppose that $j = j_t$ is the smallest integer such that $f_t^j P_t$ u-crosses R_0. Since f_t is uniformly hyperbolic outside P_t, $\{j_t\}$ has a upper bound, we again denote it by j. Suppose $f_t R_0$ is contained in $R_{i_1} \cup \cdots \cup R_{i_s}$ for some s. Then it is easy to see by invariance of measure ν_t that

$$\nu_t(R_0 \setminus P_t) \leq j \sum_{l=1}^{s} \nu_t(R_{i_l}).$$

Since

$$\nu_t\big(\cup_{i\neq 0} R_i\big) + \nu_t(R_0 \setminus P_t) = 1$$

for all $0 < t \leq 1$, we know that $\nu_t\big(\cup_{i\neq 0} R_i\big)$ is uniformly bounded away from 0 for all t. Therefore, there is $0 < c_3 < 1$ and i such that $\nu_{t_n} R_i \geq c_3$ for infinitely many n. By taking a subsequence we may think that this is true for every n. Hence we get

$$\nu_{t_n}(Q_{t_n}) \geq c_2 c_3 > 0.$$

It is a contradiction. $\square$

Put

$$P_{0,t} = P_t, \quad P_{i,t} = f_t^{-1} P_{i-1,t} \cap P_{i-1,t} \quad \text{and}$$
$$Q_{i,t} = \{x \in Q_t : f_t^j x \in P_t \text{ for } j = 1, \cdots, i\} \quad \forall i = 1, 2, \cdots .$$

Lemma 5.2. *There is $C>0$ such that for all $t<(k_0+i)^{-\beta}$,*

$$\frac{1}{C(k_0+i)^{\beta n^u}} \le \nu_t(Q_{i,t}) \le \frac{C}{(k_0+i)^{\beta n^u}}.$$

where k_0 is a real number satisfies $\delta = k_0^{-\beta}$.

Proof. By (2.4) and the definition of f_t, we know that for $t<(k_0+i)^{-\beta}$, $W^u(p,P_{i,t})$ is a ball of radius $(i+k_0)^{-\beta}$ on $W^u_\delta(p)$. So $m^u(W^u(p,P_{i,t})) = C_1(i+k_0)^{-\beta n^u}$, where C_1 is equal to the volume of the unit ball in n^u dimensional Euclidean space.

For any $x\in Q_{i,t}$, consider the holonomy that $\theta : W^u(x,Q_t,f_t)\to W^u(x,P_t)$ maps $W^u(x,Q_{i,t},f_t)$ to $W^u(p,P_{i,t})$. By Proposition 3.4, there exists $J_1^s\ge 1$ such that

$$\frac{1}{J_1^s} \le \frac{m^u(W^u(x,Q_{i,t},f_t))}{m^u(W^u(p,P_{i,t}))} \le J_1^s.$$

Now by Corollary 4.4, the ratio $\rho_{x,t}(y)/\rho_{x,t}(z)$ of the density of the conditional measure of $\nu^\xi_{x,t}$ at any two points $y,z\in W^u(x,Q_{i,t},f_t)$ are uniformly bounded by a constant $J>0$. So we get the result. □

Proof of Theorem A.

By (1.1), we only need to show that

$$\int \log|\det Df_t|_{E^u(f_t)}|d\mu_t \to 0 \quad \text{as} \quad t\to 0.$$

Take $\varepsilon>0$. Since $Df_0(p)|_{E^u_p(f_0)} = \mathrm{id}$, there is a constant $t'>0$ and a neighborhood V of p such that $\log|\det Df_t(x)|_{E^u_x(f_t)}| \le \varepsilon/2$ for all $x\in V$ and $t\in(0,t']$.

It is easy to see that there is $\delta^*>0$ such that V contains the rectangle $[W^s_{\delta^*}(p), W^u_{\delta^*}(p)]$ for all $0<t\le 1$. Take $j>0$ such that $f_t^j W^s(p,P_t)\subset W^s_{\delta^*}(p)$ for all $t\in(0,1]$. This is possible since f_t is uniformly contracting along stable direction. Also, take some $k>0$ such that $(k_0+k)^{-\beta}\le\delta^*$. Since $W^u(p,P_{k,t})\subset W^u_{\delta^*}(p)$ for all t, We have $f_t^j P_{k+j,t}\subset V$.

Note that $Q_{1,t}$ is the set of points that enter P_t under the map f_t, and $P_t\backslash P_{1,t}$ is the set of points x that leave P_t under f_t. We have $Q_{1,t}$ and $P_{1,t}$ are disjoint and

$$Q_{1,t}\cup P_{1,t} = f_t^{-1}(P_t); \quad P_t = (P_t\setminus P_{1,t})\cup P_{1,t}.$$

Since $\nu_t(f_t^{-1}P_t) = \nu_t(P_t)$, we have $\nu_t(Q_{1,t}) = \nu_t(P_t\setminus P_{1,t})$. Similarly, we have $\nu_t(Q_{i,t}) = \nu_t(P_{i-1,t}\setminus P_{i,t})$. Hence,

$$\nu_t(\Lambda_{f_t}\setminus P_{k+j,t}) = \nu_t(\Lambda_{f_t}\setminus P_t) + \sum_{i=1}^{k+j-1}\nu_t(Q_{i,t}).$$

By the definition of ν_t we know that $\nu_t(\Lambda_{f_t} \setminus P_t) = 1$ for all $0 < t \leq 1$ and by Lemma 5.2, $\sum_{i=1}^{k+j-1} \nu_t(Q_{i,t})$ is uniformly bounded for all $0 < t \leq 1$ since the selections of k, j are independent of t.

Also note that $\beta n^u < 1$. So the series $\sum_{i=1}^{\infty} (i + k_0)^{-\beta n^u}$ diverges. Therefore we can choose $N > 0$ large enough such that if all $\nu_t(Q_{i,t})$, $1 \leq i \leq N$, satisfy the estimates in Lemma 5.2, then

$$\frac{\varepsilon}{2D} \sum_{i=k+j}^{N} \nu_t(Q_{i,t}) \geq \nu(\Lambda_{f_t} \setminus P_{k+j,t}),$$

where $D = \max\{\log |\det Df_t(x)|_{E^u_x(f_t)}| : \ x \in \Lambda_{f_t}, t \in (0, 1]\}$. Then we take

$$0 < t'' \leq (N + k_0)^{-\beta}.$$

So for any $t \in (0, t'']$, the above inequality holds, and therefore,

$$\frac{\varepsilon}{2D} \nu_t(P_{k+j,t}) = \frac{\varepsilon}{2D} \sum_{i=k+j}^{\infty} \nu_t(Q_{i,t}) \geq \nu_t(\Lambda_{f_t} \setminus P_{k+j,t}).$$

By (5.1), $\mu_t = [\mu_t(\Lambda_{f_t} \setminus P_t)]^{-1} \nu_t$, the above inequality also holds for μ_t if $t \in (0, t'']$. Since

$$\mu_t(P_{k+j,t}) + \mu_t(\Lambda_{f_t} \setminus P_{k+j,t}) = 1,$$

we get $\mu_t(\Lambda_{f_t} \setminus P_{k+j,t}) \leq \dfrac{\varepsilon}{2D}$. It follows

$$\mu_t(\Lambda_{f_t} \setminus f_t^j P_{k+j,t}) \leq \frac{\varepsilon}{2D}. \tag{5.2}$$

Recall $f_t^j P_{k+j,t} \subset V$, in which $\log |\det Df_t(x)|_{E^u_x(f_t)}| \leq \varepsilon/2$ for $t \in (0, t')$. We get that if $0 < t \leq \min\{t', t''\}$, then

$$\begin{aligned}
&\int \log |\det Df_t|_{E^u(f_t)}| d\mu_t \\
= &\int_{\Lambda_t \setminus f_t^j P_{k+j,t}} \log |\det Df_t|_{E^u(f_t)}| d\mu_t + \int_{f_t^j P_{k+j,t}} \log |\det Df_t|_{E^u(f_t)}| d\mu_t \\
\leq &\ D \cdot \mu_t(\Lambda_t \setminus f_t^j P_{k+j,t}) + \frac{\varepsilon}{2} \cdot \mu_t(f_t^j P_{k+j,t}) \leq D \cdot \frac{\varepsilon}{2D} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

We proved the theorem. □

Proof of Corollary.

By the proof of Theorem A, we know that for any neighborhood V of p, there are constants $k, j > 0$ such that $f_t^j P_{k+j,t} \subset V$. Also, for any $\varepsilon > 0$ we can find $t'' > 0$ such that for all $t \in (0, t'']$, (5.2) holds. It means that $\mu_t(V) \geq \mu_t(f_t^j P_{k+j,t}) \to 1$ as $t \to 0$. This implies that $\mu_t \to \delta_p$. The rest result of the corollary is directly from Theorem A. □

Proof of Theorem B.

The proof of the homotopy follows the construction of f_t. The proof that f_0 has an infinite SRB measure is the same as in [HY] and thus omitted. □

Remark. *A referee pointed out that one can obtain Theorem A easily if nonexistence of SRB measures for f_0 is assumed. In fact, after the construction of the map f_t, one can take a weak limit of a subsequence μ_{t_i} of the SRB measures for f_{t_i} as $t_i \to 0$, and get an f_0 invariant measure μ'. Using the fact that there is a common generator ξ for all f_t such that $h_{\mu_t}(f_t) = h_{\mu_t}(f_t, \xi)$, one can show that $\limsup_{n\to 0} h_{\mu_{t_i}}(f_{t_i}) \le h_{\mu'}(f_0)$. It implies that $h_{\mu'}(f_0) \ge \int \log|\det Df_0|_{E^u}|d\mu'$. Hence, μ' satisfies entropy formula by Ruelle inequality, and therefore, if μ' has positive Lyapunov exponents, then it must be an SRB measure following a result of Ledrappier and Young (*[LY]*), which contradicts the assumption. So μ' does not have positive Lyapunov exponents, and hence $h_{\mu'}(f_0) = 0$.*

As stated in Remark 1.4, Theorem B implies that f_0 does not admits SRB measures. Thus, Theorem A can be proved by the arguments above by using Theorem B. In particular, for Anosov systems on the two dimensional torus, nonexistence of SRB measures follows from a result of Hu and Young ([LY]*).*

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