STATISTICAL PROPERTIES OF SOME ALMOST HYPERBOLIC SYSTEMS

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Abstract. Almost hyperbolic systems are smooth dynamical systems that are hyperbolic everywhere except for a finite set of points. Evidences from low-dimensional cases show that the topological properties of such systems are similar to that of uniformly hyperbolic systems, and the ergodic properties may be quite different. It admit SRB measures or infinite SRB measures. In the latter case, the systems are statistically deterministic in the sense that almost every orbit spends one hundred percent of its time arbitrarily close to the indifferent fixed points, though they are still topologically mixing. Even in the former case, correlation decay may change from exponential to power law.

0. Introduction

A smooth dynamical system is almost hyperbolic if it is hyperbolic everywhere except at a finite set of points. Ergodic properties of an almost hyperbolic system may be quite different from that of a uniformly hyperbolic system, even though we can make them topologically conjugate.

Let us imagine that we slowly deform a hyperbolic toral automorphism near the origin until the differential becomes identity. For some deformation the system does not admit any Sinai-Ruelle-Bowen measure. Instead, it admits a σ-finite measure that has absolutely continuous conditional measures on unstable manifolds. We call it an *infinite SRB measure*. For some other deformation, the resulting system still admits an SRB measure. However, the rate of decay of correlations of the system changes from exponential to polynomial. Similar phenomena also exist in some expanding maps with indifferent fixed points.

These properties imply different statistical behavior for the system. If the system has an infinite SRB measure, then for Lebesgue almost every initial condition, the orbit spends nearly one hundred percent of its time arbitrarily close to the indifferent fixed point, though it is also true that almost every orbit is dense on the torus. Hence, the system is deterministic from statistical point of view, and chaotic from topological point of view. The origin is a saddle, but statistically it looks like a sink. For the case that the system admits an SRB measure, the orbits visit every open set with positive frequency. However, they spend longer time near the indifferent fixed point, since the density of the measure is unbounded there. Therefore, the rate of mixing of the system, if measured by the rate of decay of

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correlations, is slower. So we can think that the system is "less chaotic" than the undeformed system.

The behavior of such systems mainly depends on the distribution of the physically observable measure μ in the neighborhoods of the indifferent fixed point. More precisely, it depends on how fast the measure escape from the neighborhoods. Let P_n be the sets of points that escape from a neighborhood at time n. If $\sum_{n=1}^{\infty} \mu P_n$ diverges, then the system has an infinite SRB measure, and it is statistically deterministic. If $\sum_{n=1}^{\infty} \mu P_n$ converges, but μP_n decrease in a polynomial rate, then the system has an SRB measure and has polynomial decay of correlations. This is the "less chaotic" case. If μP_n decrease exponentially fast, then the system has an SRB measure and has exponential decay of correlations. This happens in uniformly hyperbolic systems and some other nonuniformly hyperbolic systems people have studied.

Sometimes the information can be observed directly from the dynamics. Let us consider a piecewise smooth almost expanding map with an indifferent fixed point p at which $|\det Df_p| = 1$. The potential functions ϕ we are interested in (for example, $\phi = -\log|\det Df|$,) is negative and generically is of order n^{-1} on P_n. If on P_n, $\phi \sim -(\beta+1)/n$, then the absolutely continuous invariant measure μ is infinite for $0 < \beta \le 1$, and is finite with $O(n^{-(\beta-1)})$ as the rate of decay of correlations for $\beta > 1$. The systems we will discuss in Theorem 3.11, Fact 3.14 and Theorem 3.17 belong to the case. However, β may vary with x on P_n for systems in higher dimensional spaces. Therefore more detailed analysis is required.

In this paper we mainly consider almost Anosov systems on the surfaces and almost expanding maps on the intervals. We will describe the local behavior of the systems near the indifferent fixed points, and introduce the main idea for existence of SRB measures and rates of decay of correlations. Similar results should hold in general almost hyperbolic systems, however, we don't know very much beyond these maps.

1. Definitions

We start with hyperbolic case. We define the rank of a cone in $\mathbb{R}^n$ as the maximal dimension of a linear subspace contained in the cone. A pair of cones is called complete if the sum of their ranks is n.

Definition 1.1. Let f be a C^r, $r > 1$, diffeomorphism from a manifold M to itself. A closed invariant subset $\Lambda \subset M$ is called an *almost hyperbolic set*, if there exist two continuous families of complete closed cones $\mathcal{C}^u_x$ and $\mathcal{C}^s_x$ on the tangent bundle $T_\Lambda M$ such that except at a finite set $S \subset \Lambda$,

i) (invariance) $Df_x \mathcal{C}^u_x \subset \operatorname{int} \mathcal{C}^u_{fx}$, and $Df_x \mathcal{C}^s_x \supset \operatorname{int} \mathcal{C}^s_{fx}$;

ii) (hyperbolicity) $|Df_x(v)| > |v|$, $|Df^n_x(v)| \to \infty$ $\forall v \in \mathcal{C}^u_x$,
and $|Df_x(v)| < |v|$, $|Df^n_x(v)| \to 0$ $\forall v \in \mathcal{C}^s_x$.

If $\Lambda = M$, then we call the system an *almost Anosov diffeomorphism*.

It is possible that $|Df_x(v)| = |v|$ for $|v| \in \mathcal{C}^u_x$ or $\mathcal{C}^s_x$ if $x \in S$. So the systems are not uniformly hyperbolic.

The reason that we use cones rather than decomposition $TM = E^u \oplus E^s$ is that the unstable and stable bundles may not be continuous. In uniformly hyperbolic systems, continuity of $x \to \mathcal{C}_x^u$ and $\mathcal{C}_x^s$ follows from invariance and hyperbolicity. If we allow $Df_p = \mathrm{id}$ for some $p \in S$, then the cone families are not necessary continuous. In fact, the curvature of unstable and stable leaves depends on hyperbolicity. Near these points, the leaves can have a u-turn. This happens in an example constructed by A. Katok [K] of a C^∞ Bernoulli diffeomorphism on the 2-dimensional sphere that is hyperbolic everywhere except at four points. In [GK] M. Gerber and A. Katok constructed a C^∞ smooth model for pseudo-Anosov homeomorphisms in which there are more than one unstable and stable directions near the exceptional points. (See also [G] for analytic models.) These are impossible if the cone families continuous at every point in the manifolds.

One may also consider the case that S is a submanifold of M. (See [N] for a generalization of Theorem 3.8 to the case.)

For noninvertible systems, we consider piecewise almost expanding maps.

Definition 1.2. Let f be a C^r, $r > 1$, piecewise smooth map from a manifold M to itself, and $\Lambda \subset M$ be a closed invariant subset. We say that f is *almost expanding* on Λ if f is uniformly expanding away from a finite set S. That is, for any $\epsilon > 0$, there is $\kappa > 1$ such that if $x \in \Lambda \backslash B(S, \epsilon)$, then

$$|Df_x(v)| > \kappa |v| \qquad \forall v.$$

Without loss generality, we assume that in both cases S is an f-invariant set. By considering f^n, we assume that S consists of fixed points. We are interested in the case that the fixed points are indifferent, because otherwise the systems become uniformly hyperbolic or uniformly expanding. Further, we assume that S consists of only one indifferent fixed point p except when otherwise stated.

Definition 1.3. A fixed point p is called *indifferent* if Df_p has an eigenvalue λ with $|\lambda| = 1$.

In this paper we only consider the systems that are topologically transitive, and allow Markov partitions $\mathcal{R}$ into finite number of elements $\{R_j\}$. We denote $\mathcal{R}_n = \bigvee_{i=0}^{n-1} f^{-i}\mathcal{R}$, and then let $\mathcal{R}_n(x) \in \mathcal{R}_n$ be the element that contains x. Restricted to each element, f is C^r with $r > 1$. So the systems are smooth or piecewise smooth. We also assume that R_0 is the element containing p. We denote

$$P_n = \{x \in R_0 : f^k x \in R_0, 0 \le k \le n-1, f^n x \notin R_0\}, \tag{1.1}$$

$Q_n = \bigcup_{i=0}^{n} P_i$ and $\mathcal{O}_n = \bigcup_{i=n}^{\infty} P_i$. We will regard $P_0 = \Lambda \backslash R_0$

We will use the notation "$\sim$" very often, where $a(n) \sim b(n)$ means that $a(n) = b(n) + O(n^{-(1+t)})$ for some $t > 0$.

2. Dynamics near Indifferent Fixed Points

Consider an almost expanding map f on the unit interval I. Assume that 0 is the fixed point and I_0 is the element of the Markov partition that contains 0.

Take $x \in I_0$. Consider the backwards orbit $\{x_n\}_{n=0}^\infty$ of x, where $x_n = f^{-n}x \cap I_0$. We know that if $f'(0) > 1$, then x_n converges to 0 exponentially fast. In our case, the rate is polynomial.

Fact 2.1. *Suppose*

$$fx = x\big(1 + x^\gamma + o(x^\gamma)\big), \qquad x \to 0. \tag{2.1}$$

Then for any $x \in I_0$, the backwards orbit $\{x_n\}_0^\infty$ satisfies $x_n^\gamma \sim (\gamma n)^{-1}$.

Remark 2.2. By the fact we can see that on P_n,

$$f'x \sim 1 + \frac{1+\beta}{n}, \qquad \beta = 1/\gamma. \tag{2.2}$$

We cannot expect bounded distortion on any interval that contains 0, since $(f^n)'(0) = 1$ and $(f^n)'(x_n)$ is unbounded by Fact 2.1 and (2.2). However, the following fact says that restricted to P_k, distortion is bounded. This is enough for us to get absolute continuity of invariant measures and to estimate the rates of decay of correlations.

Proposition 2.3. *There exists constant $J > 0$ such that for any $x, y \in P_k$, for any $n > 0$, if $x_n \in f^{-n}x$ and $y_n \in f^{-n}y \cap \mathcal{R}_n(x_n)$, then*

$$\log \frac{\big|(f^n)'(y_n)\big|}{\big|(f^n)'(x_n)\big|} \le \frac{J}{x} d(x, y).$$

Now we consider almost hyperbolic systems on surfaces. Let p be an indifferent fixed point. If one of eigenvalues of Df_p is not equal to 1, then there is a gap between the spectrum of Df. So we can apply the stable manifold theorem ([HPS]) to obtain unstable and stable manifolds, which form C^1 foliations. The latter means that the holonomy maps by sliding along the unstable or stable leaves are C^1. So near p, by choosing a suitable coordinate system, the map can be written as

$$f(x, y) = \big(x + ax^{2k+1} + \text{h.o.t.},\ \lambda y + \text{h.o.t.}\ \big), \tag{2.3}$$

for some $\lambda > 1$, $a < 0$, or $\lambda < 1$, $a > 0$. Hence the orbits change polynomially in one direction and exponentially in the other direction. (Note that we need odd power for the second term in x-component to keep the map hyperbolic.)

For the case that 1 is the only eigenvalues of Df_p, there are two possibilities: the eigenspace of Df_p is one or two dimensional. In the first case the unstable and stable manifolds are tangent at p. These maps were studied by J. Lewowicz ([Lo]). Recently, it is proved by E. Catsigeras and H. Enrich [CE] that with some conditions on the coefficient of the Taylor expression, f have an SRB measure.

In the second case, which is simpler and has been understood better, $Df_p = \text{id}$. We assume the systems satisfy nondegenerate conditions below. It seems that similar results are also true if we use higher order jet instead of the third order one. However, there is no reference about it.

Definition 2.4. An almost Anosov diffeomorphism $f : M \to M$ is said to be *nondegenerate*, if there are constants $\kappa^+, \kappa^- > 0$ such that for all x near p,

$$\begin{aligned} |Df_x v| &\ge \big(1 + \kappa^+ d(x,p)^2\big)|v| \qquad \forall v \in \mathcal{C}_x^u, \\ |Df_x v| &\le \big(1 - \kappa^- d(x,p)^2\big)|v| \qquad \forall v \in \mathcal{C}_x^s\ . \end{aligned}$$

Note that $Df_p = \text{id}$ by the assumption, $D^2 f_p = 0$ by hyperbolicity, and $D^3 f_p \ne 0$ by the nondegenerate conditions. So near p the motion is dominated by $D^3 f_p$, which makes the dynamics quite different (see [H1] for more details).

Since $D^3 f_p$ is a 3-form, the immediate consequence is that near p, f^{n^2} acts on x/n just like f acting on x up to a scale of n times. That is, $f^{n^2}(tx/n) \sim f(tx)/n$, or more precisely, $\lim_{t\to 0} \dfrac{f^{n^2}(tx/n)}{f(tx)/n} = 1$. Df^{n^2} has the same property. We should recall that near a hyperbolic fixed point, the corresponding relation is $f(tx/n) \sim f(tx)/n$, since f is close to a hyperbolic linear map. The facts are very helpful for us to understand the behavior of orbits when they are near p.

The system allows a decomposition of the tangent bundle into $TM = E^u \oplus E^s$. For $x \neq p$, existence and continuity of E^u_x and E^s_x can be obtained in the same way as for uniformly hyperbolic systems. Further, E^u_x and E^s_x satisfy the Hölder condition outside any neighborhood of p. (The Hölder coefficient increases as the size of the neighborhood shrinks.) We can also define E^u_p and E^s_p: they are the only lines such that $D^3 f_p(v, v, v)$ and v collinear for any v in the line. From the similarity between Df at x and Df^{n^2} at x/n, we can see that for any x, E^u_{tx} and E^s_{tx} has a limit as $t \to 0$, and the limits are different if the directions of x are different. This is because near p, $Df_x = \mathrm{id} + (1/2)D^3 f(x, x, \cdot)$, and $D^3 f(x, x, \cdot)$ change as the direction of x change, while near a hyperbolic fixed point, all Df_x are close to Df_p.

Note that between unstable and stable parts, there is no gap in the spectrum of Df. The stable manifold theorem doesn't apply. However, (weak) unstable and stable manifolds can be obtained by graph transformation for $x \neq p$. $W^u_{\mathrm{loc}}(p)$ and $W^s_{\mathrm{loc}}(p)$ can be defined as the set of points whose backward or forward orbits stay inside a small neighborhood of p. They are smooth curves and tangent to E^u_p and E^s_p respectively.

We have similar distortion estimates as for one dimensional case. The following result is proved in [H1] and refined in [H4].

Let P be a rectangle with $p \in \operatorname{int} P$. Recall the definition of P_n in (1.1).

Proposition 2.5. *There exists constant $I > 0$, depending on P, $\theta \in (0, 1)$ such that for any $x, y \in P_k$, $k \geq 0$, with $y \in W^u_{\mathrm{loc}}(x)$,*

$$\log \frac{|Df_y^{-n}|_{E^u_y}|}{|Df_x^{-n}|_{E^u_x}|} \leq I d^u(x, y)^\theta \qquad \forall n > 0.$$

Away from p, the unstable and stable foliations are Lipschitz. That is, the holonomy maps $\pi_{xy} : W^{u(s)}_{\mathrm{loc}}(x) \to W^{u(s)}_{\mathrm{loc}}(y)$ are Lipschitz, if they stay away from $W^u_{\mathrm{loc}}(p)$ and $W^s_{\mathrm{loc}}(p)$. (Of cause, the Lipschitz constants increase as they get closer to p.) On the other hand, for any x close to p, the holonomy maps from $W^u_{\mathrm{loc}}(x)$ to $W^u_{\mathrm{loc}}(p)$ is only Hölder. Let us assume that in a neighborhood of p the map f can be written as the following in a local coordinate system,

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + x\big(ax^2 + by^2 + O((x^2 + y^2)^{3/2})\big) \\ y - y\big(cx^2 + dy^2 + O((x^2 + y^2)^{3/2})\big) \end{pmatrix}. \tag{2.4}$$

Note that for any $z \in W^u_{\mathrm{loc}}(p)$, $d(p, f^{-n}z)$ have the order $n^{-1/2}$ by Fact 2.1.

Proposition 2.6. *Let $q \in W^s_{\mathrm{loc}}(p)$ with $q \neq p$. Then for any $z \in W^u_{\mathrm{loc}}(p)$, $d\big(q, \pi_{pq}(f^{-n}z)\big) = O(n^{-b/2d})$.*

The non-Lipschitzness of the stable foliation is very important for the statistical behavior of the systems. It is $d\big(q, \pi_{pq}(f^{-n}z)\big)$, not $d(p, f^{-n}z)$, that plays the role, as we will see in Proposition 3.4 and Theorem 4.5. (If we construct a tower as in [Y1] or [Y2], then $d\big(q, \pi_{pq}(f^{-n}z)\big)$ is the size of the n'th level.)

3. Invariant Measures

3.1. **SRB Measures and Infinite SRB Measures.** For an Anosov system $f : M \to M$, a result of Sinai (see e.g. [S]) says that f admits a unique invariant

Borel probability measure μ with the property that μ has absolutely continuous conditional measures on unstable manifolds. This is the invariant measure that is observed physically, for if $\phi : M \to \mathbb{R}$ is a continuous function, then for Lebesgue almost every point $x \in M$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) = \int \phi d\mu.$$

These results have been extended to Axiom A attractors by Ruelle, Bowen, etc. (See e.g. [B].) This measure is called an Sinai-Ruelle-Bowen measure or an SRB measure. Now there are some results about existence of SRB measures for nonuniformly hyperbolic systems such as Hénon maps and hyperbolic systems with singularities. (See [BY], [P], [JN], [BV], etc.)

Definition 3.1. An f-invariant probability measure μ is called an *SRB measure* for $f : \Lambda \to \Lambda$ if

i) (f, μ) has positive Lyapunov exponents almost everywhere;
ii) μ has absolutely continuous conditional measures on unstable manifolds.

For the precise meaning of the second condition above, we refer [LS].

Some almost Anosov systems do not admit SRB measures. Instead, they admit σ-finite measures that have similar properties.

Definition 3.2. An f-invariant σ-finite measure μ is called an *infinite SRB measure* for $f : \Lambda \to \Lambda$ if $\mu\Lambda = \infty$, and for any $R \subset \Lambda$ with $\mu(R) < \infty$ and $R = \overline{\operatorname{int} R}$,

i) (f_R, μ_R) has positive Lyapunov exponents almost everywhere, and
ii) μ_R has absolutely continuous conditional measures on unstable manifolds of f.

where f_R is the first return map of f on R, and μ_R is the conditional measure of μ on R.

In this paper, the term "SBR measure" without any qualifications will be reserved for probability measures.

We will see in Corollary 3.5 that orbit distributions are quite different between systems with SRB measures and those with infinite SRB measures.

For almost Anosov systems on compact surfaces, we have the following.

Theorem 3.3. *Every nondegenerate almost Anosov diffeomorphism on a surface has either an SRB measure or an infinite SRB measure.*

Proof. Take a suitable rectangle P that contains p as its interior point. Let $g : M\backslash P \to M\backslash P$ be the first return map.

Take a piece of unstable curve γ. Let m_γ be the Lebesgue measure on γ and let $g_*^n m_\gamma$ be the push-forward of m_γ, i.e., $(g_*^n m_\gamma)(E) = m_\gamma(g^{-n}E)$. Then any accumulation point $\bar{\mu}$ of $n^{-1} \sum\limits_{i=0}^{n-1} g_*^i m_\gamma$ in the weak* topology is a g-invariant measure.

Let ρ_i denote the density of $g_*^i m_\gamma$ with respect to the Lebesgue measure on $g^i\gamma$. Then by distortion estimates in Proposition 2.5 for any small rectangle $R \subset M\backslash P$, for all x, y in a same component of $g^i\gamma \cap R$ that crosses R,

$$e^{-I} \le \frac{\rho_i(x)}{\rho_i(y)} \le e^I. \tag{3.1}$$

This bound on ρ_i is passed on to the limit measure $\bar{\mu}$. Then we get absolute continuity.

We can extend $\bar{\mu}$ to an f-invariant measure μ in an obvious way. If μ is finite, then we get an SRB measure (after normalization), otherwise we get an infinite SRB measure. $\square$

To determine if μ is finite, we only need check if the return time on $f^{-1}P\backslash P$ is integrable, or equivalently, to see if μP is finite. The following proposition gives a direct way to estimate it, in which the notations are the same as in Proposition 2.6.

Proposition 3.4. *Let $q \in W^s_{\text{loc}}(p)$ with $q \in f^{-1}P\backslash P$. Then μ is finite if and only if $\sum_{n=1}^{\infty} d(q, \pi_{pq}(f^n z))$ is convergent.*

Proof. Let $P'_n = \{x \notin P : f^i x \in P,\ 1 \le i \le n\}$. We have $\mu P = \sum_{n=1}^{\infty} \bar{\mu} P'_n$ since $P = \bigcup_{n=1}^{\infty} f^n P'_n$. $\mu P'_n$ is proportional to the length $l(\gamma^u_n)$ of any unstable curve $\gamma^u_n \subset P'_n$ that crosses P'_n, because (3.1) also holds for the density function of the conditional measures of μ. Now we use the fact that $l(\gamma^u_n)$ is proportional to $d(q, \pi_{pq}(f^n z))$. $\square$

Corollary 3.5. *Suppose $f : M \to M$ is an almost Anosov diffeomorphism with only one indifferent fixed point p. Let $\phi : M \to \mathbb{R}$ be a continuous function.*

If f admits an SRB measure μ, then for Leb-a.e. $x \in M$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) = \int \phi d\mu.$$

If f admits an infinite SRB measure μ, then for Leb-a.e. $x \in M$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) = \phi(p).$$

The second part of the corollary is true because the measure of any neighborhood of p is infinite, while the measure of its complement is finite.

As we mentioned before that if f admits an infinite SRB measure, then it is statistically deterministic and topologically chaotic. Here is another feature.

Corollary 3.6. *If f has an infinite SRB measure, then the forward Lyapunov exponents are 0 for Leb-a.e. $x \in M$.*

Proof. Take $\phi(x) = \log |Df_x|_{E^u_x}|$ and $\log |Df_x|_{E^s_x}|$ in the above corollary. Continuity of ϕ follows from continuity of E^u_x and E^s_x for $x \neq p$ and the fact $Df_x \to Df_p = \text{id}$ as $x \to p$. $\square$

It seems that whether a system has an SRB measure depends on whether expansion is stronger than contraction near the indifferent fixed point. Here we discuss some special cases. Assume that Df_p has eigenvalues $\overline{\lambda} \ge \underline{\lambda} > 0$.

Fact 3.7. *If $\overline{\lambda} > 1$, then f has an SRB measure.*

This is because the system is uniformly expanding along unstable directions. So distortion for $Df|_{E^u}$ is bounded. Therefore the same "push-forward" method gives an probability invariant measure.

If $\underline{\lambda} < 1$ besides $\overline{\lambda} > 1$, then the system is an Anosov system.

Theorem 3.8 ([HY]). *If $\overline{\lambda} = 1$, $\underline{\lambda} < 1$, then f has an infinite SRB measure.*

Proof. By (2.3) and Fact 2.1, $d(p, f^{-n}z)$ is of order $n^{-1/2k}$ $\forall z \in W^u_{\text{loc}}(p)$. Then the result follows from Proposition 3.4 and the fact that the stable foliation is C^1. $\square$

If $Df_p = \text{id}$, then we need look at the third order terms of the Taylor expression of the map at p. We only have some sufficient conditions to guarantee that both cases in Theorem 3.3 do occur. Assume that we can choose a suitable coordinate system such that f can be written as

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + x\big(ax^2 + rxy + by^2 + O((x^2+y^2)^{3/2})\,\big) \\ y - y\big(cx^2 + sxy + dy^2 + O((x^2+y^2)^{3/2})\,\big) \end{pmatrix}.$$

Theorem 3.9 ([H1]). *With the above notations,*

i) *f has an SRB measure if $b > 2d$, $r = 0 = s$, and $ad > bc$;*
ii) *f has an infinite SRB measure if $2b < d$ and $rs \neq 0$.*

Proof. Part i) follows from Proposition 2.6 and 3.4. For Part ii), we can proof that $d\big(q, \pi_{pq}(f^{-n}z)\big)$ is bounded below by $O(n^{-1/2})$. $\square$

The conditions in the first part say $a/c > b/d > 2$. So expansion is stronger. For the second part, we can see that at least there is a cone on the plane in which contraction is stronger.

If the only eigenvalue of Df_p is 1 and $Df_p \neq \text{id}$, then near p, f has the form

$$f(x, y) = (x + ay + r(x, y), y + s(x, y)),$$

where $a \neq 0$, and r and s consist of higher order terms of x and y.

Theorem 3.10 ([CE]). *If the coefficient of x^2 in r is zero and coefficient of x^3 in s is nonzero, then f has an SRB measure.*

3.2. **Absolutely Continuous Invariant Measures.** It is well-known that piecewise smooth uniformly expanding systems on the unit interval have absolutely continuous invariant measures. (A folklore theorem, also see e.g. [KS], [W] for higher dimensional results.) For almost expanding maps on the unit interval, it is proved by G. Pianigiani [Pi] that absolutely continuous invariant measures exist, which may be finite or σ-finite.

The proof in [Pi] uses the first return maps. Here we give an alternative one. This method can be easily adopted to a more general setting.

Theorem 3.11. *Let $f : I \to I$ be a piecewise smooth expanding map with an indifferent fixed point 0. Assume that near 0, f satisfies (2.1). Then f has an absolutely continuous invariant measure μ, and μ is finite if $0 < \gamma < 1$, σ-finite if $\gamma \geq 1$.*

Proof. Take $\phi(x) = -\log f'(x)$. Define the Perron-Frobenius operator by

$$\mathcal{L}_\phi g(x) = \sum_{fx_1 = x} e^{\phi(x_1)} g(x_1).$$

Since the fixed point of $\mathcal{L}_\phi$ is unbounded near 0, we take a Banach space $\mathcal{B} = C^0(0,1]$ but with the norm $\|g\| = \sup\{x^{2\gamma J} g(x) : x \in I\}$. Then we construct a convex set

$$\begin{aligned} \mathcal{G} = \{g \in \mathcal{B} : \quad & g > 0,\ g(1) = 1,\ \exists H_0 \text{ s.t. } x^\gamma g(x) \le H_0\ \forall x \in I_0, \\ & g(y) \le g(x)(1 + Jx^{-1} d(x,y)\ \forall x, y \in P_n\}. \end{aligned} \tag{3.2}$$

Note that as $x \to 0$, $x^{2\gamma J} g(x) \to 0$ uniformly for all $g \in \mathcal{G}$. So $\mathcal{G}$ is compact in $\mathcal{B}$.

Let $\tilde{\mathcal{L}}_\phi g = \mathcal{L}_\phi g / \mathcal{L}_\phi g(1)$. Clearly, $\tilde{\mathcal{L}}_\phi$ is continuous. It can be proved that $\tilde{\mathcal{L}}_\phi \mathcal{G} \subset \mathcal{G}$. So by Schauder-Tychonoff fixed point theorem, $\tilde{\mathcal{L}}_\phi$ has a fixed point $h \in \mathcal{G}$. Hence, $\mathcal{L}_\phi h = h$. Using h as the density function, we get an invariant measure μ. By the corollary below, h is integrable if and only if $\gamma < 1$. □

The following result tells that on P_n, $h(x)$ increases like $\gamma n \tilde{h}(0)$, where $\tilde{h}(x) = \sum\limits_{\hat{x}_1 \in f^{-1}x \setminus I_0} \frac{h(\hat{x}_1)}{f'(\hat{x}_1)}$. (Recall Fact 2.1.)

Corollary 3.12. $\lim\limits_{x \to 0} x^\gamma h(x) = \tilde{h}(0)$.

Proof. Observe that $\mathcal{L}_\phi h = h$ implies that for $x \in I_0$, if $x_1 = f^{-1}x \cap I_0$, then

$$x_1^\gamma h(x_1) \sim x^\gamma h(x) \frac{x_1^\gamma}{x^\gamma} \cdot f'(x_1) - \tilde{h}(0) x_1^\gamma f'(x_1),$$

where $\sim$ means that the difference between the two sides is of order higher than x^γ. Since $f'(x) \sim 1 + (1+\gamma)x^\gamma$ and $x^\gamma / x_1^\gamma \sim 1 + \gamma x_1^\gamma$, we have

$$x_1^\gamma h(x_1) \sim x^\gamma h(x)(1 + x_1^\gamma) - \tilde{h}(0) x_1^\gamma.$$

So if there is $c > 0$ such that $x^\gamma h(x) \ge \tilde{h}(0)(1+c)$ for a small x, then

$$x_1^\gamma h(x_1) \ge \tilde{h}(0)(1 + c + 0.5 c x_1^\gamma)$$

because $(1+c)(1+x_1^\gamma) - x_1^\gamma = 1 + c + c x_1^\gamma$. Inductively, $x_n^\gamma h(x_n) \ge h(0)(1 + c + 0.5c \sum_{i=1}^n x_i^\gamma)$. Since $\sum_{i=1}^n x_i^\gamma$ diverges, it contradicts the fact that $x^\gamma h(x) \le H_0$ for all $x \in I_0$. So the upper limit is bounded by $\tilde{h}(0)$. The lower limit can be proved similarly. □

By checking the proof carefully, we can actually get the following.

Theorem 3.13. *Let $f : \Lambda \to \Lambda$ be a piecewise smooth almost expanding map with an indifferent fixed point. Suppose $\phi = -\log|\det Df_x|$ satisfies the following.*

i) *There is $J_\phi > 0$ such that $S_n\phi(y_n) - S_n\phi(x_n) \le J_\phi x^{-1} d(x,y) \quad \forall x, y \in P_n, x_n \in f^{-n}x, y_n \in f^{-n}y \cap \mathcal{R}_n(x_n)$.*

ii) *On P_n, $\phi(x) \le -\dfrac{\beta+1}{n}$ for some $\beta > 0$.*

Then f has an absolutely continuous invariant measure μ. Further, μ is finite if $\beta > 1$, and σ-finite if moreover $\phi(x) \ge -\dfrac{\beta'+1}{n}$ on P_n for some $0 < \beta' \le 1$.

Theorem 3.11 says that we can use γ to determine finiteness of absolutely continuous invariant measures of almost expanding systems on intervals. It is unclear what the higher dimensional versions are. However, in some simple cases, we can get similar results, which can be regarded as an application of Theorem 3.13.

Assume that an almost expanding map $f : I^m \to I^m$ has the form

$$f(x) = x(1 + |x|^\gamma) + o(x^{1+\gamma})$$

as x near 0. That is, f is a map that expands at the same rate in all directions at 0, plus a higher order perturbation. Then with the same methods, we have:

Fact 3.14. *f has an absolutely continuous invariant measure μ, which is finite if $0 < \gamma < m$, and σ-finite if $\gamma \geq m$.*

Proof. Check that on P_n, $\phi \sim -\dfrac{1 + m/\gamma}{n}$. This also implies the first condition in the above theorem. □

However, in higher dimensional spaces, ϕ may varies, say, from $-(1 + \beta_1)/n$ to $-(1 + \beta_2)/n$, $\beta_1 \neq \beta_2$, on P_n. If so, the first condition in Theorem 3.13 usually fails. In this case we need modify the proof. Here is an example.

Fact 3.15. *Suppose $f : I^2 \to I^2$ is an almost expanding map and near 0,*

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_1^{1+\gamma_1} + o(x_1^{1+\gamma_1}) \\ x_2 + x_2^{1+\gamma_2} + o(x_2^{1+\gamma_2}) \end{pmatrix}.$$

Then f has an absolutely continuous invariant measure μ, which is finite if $\gamma_1^{-1} + \gamma_2^{-1} > 1$, and σ-finite if $\gamma_1^{-1} + \gamma_2^{-1} \leq 1$.

Proof. We can partition each P_n into countable many smaller pieces, such that on each of them the map has bounded distortion. To make the set $\mathcal{G}$ compact, we use $x_1^{2\gamma_1 J} x_2^{2\gamma_2 J} g(x_1, x_2)$ to define the norm in the space $\mathcal{B}$. To compute integral of return time, we compute the sum of the area of $\mathcal{O}_n$, which are rectangles of size approximately $(\gamma_1 n)^{-1/\gamma_1} \times (\gamma_2 n)^{-1/\gamma_2}$ by Fact 2.1. □

If an almost expanding system has an absolutely continuous invariant measure μ, and it contains only one indifferent fixed point p, then a statement similar in Corollary 3.5 can be made. That is, for Lebesgue almost every point, the time average is equal to $\int \phi d\mu$ if μ is finite, and is equal to $\phi(p)$ if μ is σ-finite.

If f has two indifferent fixed points that cause σ-finite measure, then from the case of one dimensional systems, the limits for time average may not exist.

Theorem 3.16 ([I]). *Suppose f is a piecewise smooth expanding map on the interval with indifferent fixed points p and q such that*

$$|f(x) - x| = a|x - p|^{1+\alpha} + o(|x - p|^{1+\alpha}), \qquad x \to p;$$
$$|f(x) - x| = b|x - q|^{1+\beta} + o(|x - q|^{1+\beta}), \qquad x \to q.$$

Then for any small neighborhoods U and V of p and q respectively, Leb-a.e. x,

$$\lim_{n\to\infty} \frac{\sum_{i=0}^n 1_U\big(f^i(x)\big)}{\sum_{i=0}^n 1_V\big(f^i(x)\big)} = \begin{cases} 0 & \text{if } \beta > \alpha \geq 1, \\ c > 0 & \text{if } \beta = \alpha = 1, \\ \text{doesn't exist} & \text{otherwise.} \end{cases}$$

3.3. **Measures of Full Hausdorff Dimensions.** A measure ν on a bounded subset Λ in $\mathbb{R}^n$ is called an measure of full Hausdorff dimension if $HD(\nu) = HD(\Lambda)$, where $HD(\Lambda)$ is the Hausdorff dimension of Λ, and

$$HD(\nu) = \inf\{HD(\Lambda_0) : \Lambda_0 \subset \Lambda,\ \nu(\Lambda \backslash \Lambda_0) = 0\}.$$

If Λ is an invariant set of a conformal expanding map, then f has an invariant probability measure μ such that $HD(\mu) = HD(\Lambda)$. However, for an invariant set of an almost expanding map, the measure of full Hausdorff dimension may be σ-finite.

Theorem 3.17 ([U]). *Let $0 \in \Lambda \subset I$ be an invariant set of a piecewise smooth almost expanding map f with $\delta = HD(\Lambda)$. Assume that near 0, f satisfies (2.1). Then f has an invariant measure μ of full Hausdorff dimension, and μ is finite if $\delta > 2\gamma/(1+\gamma)$; μ is σ-finite if $\delta \le 2\gamma/(1+\gamma)$.*

Proof. Take $\phi(x) = -\delta \log f'(x)$. On P_n, we have $\phi(x) \sim \delta(1+\gamma^{-1})/n$. Then we use a similar method for Theorem 3.13. The role of the Lebesgue measure is replaced by a conformal measure ν which is a fixed point of the dual operator of $\mathcal{L}_\phi$ on the space $C^0(\Lambda)$. $\square$

By works of M. Urbański *et al*, Λ differ from hyperbolic Cantor sets in many properties. (See [U] for details and related references.) HD(Λ) is always bounded below by $\gamma/(1+\gamma)$, while the Hausdorff dimensions of hyperbolic Cantor sets can be arbitrary small. The topological pressure $P(f|_\Lambda, -t\log f')$ is nonnegative. (This is because near 0, $e^{S_n\phi(x_n)}$ decreases in a polynomial rate.) The pressure is positive if $t <$ HD(Λ), and 0 if $t \ge$ HD(Λ). That is, HD(Λ) is the smallest solution of $P(f|_\Lambda, -t\log f') = 0$. For hyperbolic Cantor sets, $P(f|_\Lambda, -t\log f')$, as a function of t, is strictly decreasing, and HD(Λ) is the unique solution of the equation.

3.4. **Weak Gibbs States and Equilibrium States.** We cannot expect the measures obtained in Section 3.1- 3.3 to be Gibbs States for the potential functions. This is because at an indifferent fixed point p, we may have $e^{S_n\phi(p)} = 1$ for any n, and $\mu\mathcal{O}_n \to 0$ in a polynomial rate. Therefore there is no uniform bound for the ratios between $\mu\mathcal{R}_n(p)$ and $\exp\{-nP + S_n\phi(p)\}$. Instead, we consider week Gibbs states.

The notion of weak Gibbs state was introduced by R. Dobrushin, and then studied by C. Maes *et al* (see e.g. [MRM] for related references). The definition we use below is from [Yu2]. Recall that $\mathcal{R}_n(x)$ was defined in Section 1.

Definition 3.18. A Borel probability measure μ is called a *weak Gibbs state* for a function ϕ with constant P if there is a sequence $\{C_n\}$ of positive number with $\lim\limits_{n\to\infty}(1/n)\log C_n = 0$ such that for μ-a.e. x,

$$C_n^{-1} \le \frac{\mu\mathcal{R}_n(x)}{\exp\{-nP + S_n\phi(x)\}} \le C_n. \tag{3.3}$$

It is proved in [Yu2] and [MRTMV] that absolute continuous invariant probability measures for systems discussed in Theorem 3.11 are week Gibbs states. Actually, all probability measures we discussed in previous sections are weak Gibbs states.

A weak Gibbs state of ϕ is an equilibrium state of the function, and the constant P is the topological pressure. In fact, since μ is a probability measure, (3.3) gives

$$C_n^{-1} \le e^{-nP}\sum e^{S_n\phi(x_B)} \le C_n,$$

where the sum is taken over all elements $B \in \mathcal{R}_n$ and x_B is any point in B. By the definition, we know that P is the topological pressure. Also, by (3.3),

$$\frac{1}{n}S_n\phi(x) - \frac{1}{n}\log C_n \le P + \frac{1}{n}\log \mu\mathcal{R}_n(x) \le \frac{1}{n}S_n\phi(x) + \frac{1}{n}\log C_n.$$

Since $\lim\limits_{n\to\infty}(1/n)S_n\phi(x) = \int \phi d\mu$ and $\lim\limits_{n\to\infty} -(1/n)\log\mathcal{R}_n(x) = h_\mu(\sigma)$ for μ-a.e.x by the Birkhoff ergodic theorem and the Shannon-McMillan-Breiman theorem respectively, the inequality implies $P = \int \phi d\mu + h_\mu(\sigma)$.

It is clear that equilibrium states are not unique for almost hyperbolic systems since obviously the point mass δ_p is another one.

4. Convergence Rate

We discuss decay rates of correlations and convergence rates of functions under the Perron-Frobenius operators.

4.1. **Decay of Correlations.**

The correlation of two functions $g, \hat{g} : M \to \mathbb{R}$ is

$$\rho_{g,\hat{g}}(n) = \int g \cdot (\hat{g} \circ f^n) d\mu - \int g d\mu \int \hat{g} d\mu.$$

If a system (f, μ) is mixing, then for any $g, \hat{g} \in L^2(\mu)$, $\lim\limits_{n\to 0}\rho_{g,\hat{g}}(n) = 0$. So we can think that the speed of the convergence reflects the speed of mixing. (We refer the paper by V. Baladi in this volume for general information and related references about decay of correlations.)

One of important characters for almost hyperbolic systems is that they can have polynomial decay of correlations.

Definition 4.1. We say that (f, μ) has *polynomial decay of correlations* with degree α for functions in $\mathcal{F}$, if for all $g, \hat{g} \in \mathcal{F}$, $\exists C = C(g, \hat{g}) > 0$ s.t.

$$|\rho_{g,\hat{g}}(n)| \le Cn^{-\alpha} \qquad \forall n \ge 1.$$

Polynomial decay of correlations was first proved in 1993 by Mori ([M]) and by Lambert-Siboni-Vaienti ([LaSV]) for piecewise linear maps on the unit interval with an indifferent fixed point (Takahashi model). For piecewise smooth almost expanding maps, the result was proved by Young [Y2], Liverani-Saussol-Vaienti ([LiSV]), Pollicott-Yuri ([PY]), and the author [H2], all with different approaches. The lower bounds of the decay rates for the systems are also polynomial. (See Theorem 4.3.) For invertible systems, polynomial decay is proved for some almost Anosov systems on surfaces (Theorem 4.5).

Applying the results in ([Y2]) to an almost hyperbolic system, we get that decay rates of correlations are determined by $m\mathcal{O}_n$, where m is a reference measure. Here we introduce slightly different conditions, which is given in terms of invariant measures ([H3]).

Let f be an almost expanding map preserving an invariant measure μ. Suppose $\mu \circ f << \mu$. Put

$$\psi(x) = -\log\frac{d\mu \circ f}{d\mu}(x).$$

If μ is an absolutely continuous measure with density function h, then it is easy to check that $\psi(x) = -\log\big(|\det Df_x| \cdot h(fx)/h(x)\big)$.

Let
$$\mathcal{L}_\psi g = \sum_{x_1 \in f^{-1}x} e^{\psi(x_1)} g(x_1)$$
be the Perron-Frobenius operator. Since μ is an f-invariant measure, $\mathcal{L}_\psi c = c$ for any constant c and $\mu(\mathcal{L}_\psi g) = \mu(g)$ for any integrable function g.

Theorem 4.2 ([H3]). *Suppose ψ satisfies the following:*

i) *there is $J_\psi > 0$, $\theta \in (0,1)$ such that $S_n\psi(y_n) - S_n\psi(x_n) \le J_\psi d(x,y)^\theta$ $\forall x, y \in P_n$, where $x_n \in f^{-n}x$, $y_n \in f^{-n}y \cap \mathcal{R}_n(x_n)$;*

ii) *on P_n, $\psi(x) \le -\dfrac{\beta}{n} + O\big(\dfrac{1}{n^{1+r}}\big)$, $r > 0$, for some $\beta > 1$, or*

ii') *$\forall k > 0$, $\exists C_\psi > 0$ such that $e^{S_n\psi(x_n)} \le C_\psi (n+k)^{-\beta}$, $\forall x \in P_k$.*

Then (f, μ) has polynomial decay of correlations with degree $\beta - 1$.

Clearly, condition ii) implies ii'). The latter implies $\mu\mathcal{O}_n = O(n^{-\beta})$.

Proof of Theorem 4.2: It follows directly from Theorem 4.6 in the next subsection, because $\rho_{g,\hat g}(n) = \int (\hat g \circ f^n)\cdot(g - \mu(g))d\mu = \int \hat g \cdot (\mathcal{L}_\psi^n g - \mu(g))d\mu$, where we use the fact that $\mathcal{L}_\psi$ is the dual of the operator $F : L^2(\mu) \to L^2(\mu)$ defined by $Fg(x) = g(fx)$. $\square$

For one dimensional systems, we have the following, where $\beta = 1/\alpha$.

Theorem 4.3. *Let f be a piecewise smooth expanding map on the interval with an indifferent fixed point* 0. *Assume that f satisfies (2.1) with $0 < \gamma < 1$. Then*

i) ([Y2], [H2]) *for all Lipschitz functions g, $\hat g$, there exists $C = C(g, \hat g)$ s.t.*
$$\left|\rho_{g,\hat g}(n)\right| \le Cn^{-(\beta-1)} \quad \forall n \ge 1;$$

ii) ([H2]) *there exist C^∞ functions g, $\hat g$ and a constant $C' > 0$ s.t.*
$$\left|\rho_{g,\hat g}(n)\right| \ge C'n^{-(\beta-1)} \quad \forall n \ge 1.$$

Proof of Part i): Distortion estimates for ψ follow from that for ϕ and the fact that $h \in \mathcal{G}$, defined in (3.2). For the second condition, we consider $x \in P_n$. We know $\phi(x) \sim -\dfrac{\beta+1}{n}$ by Fact 2.1. By Corollary 3.12, $\dfrac{h(fx)}{h(x)} \sim \dfrac{n-1}{n}$. So $\psi(x) \sim -\dfrac{\beta+1}{n} + \dfrac{1}{n} = -\dfrac{\beta}{n}$. $\square$

The proof for Part ii) needs different method. For lower bounds in general case, we have:

Conjecture 4.4. *Under the circumstances of Theorem 4.2, if moreover $\psi(x) \ge -\dfrac{\beta'}{n} + O\big(\dfrac{1}{n^{1+r}}\big)$, $r > 0$, for some $\beta' \ge \beta$, then the lower bounds of decay rates of the system are of order $O(n^{-(\beta'-1)})$.*

The systems discussed in Fact 3.14 and Theorem 3.17 are also have polynomial decay of correlations with the degrees $m\gamma^{-1} - 1$ and $\delta(1 + \gamma^{-1}) - 1$ respectively, whenever the measures we discussed are finite.

Pollicott and Yuri ([PY]) studied decay rate of correlations of the inhomogeneous Diophantine approximation transformation, defined by
$$f(x,y) = \Big(\frac{1}{x} - \big[\frac{1-y}{x}\big] + \big[-\frac{y}{x}\big], -\big[-\frac{y}{x}\big] - \frac{y}{x}\Big),$$

where $0 \le y \le 1$, $-1 \le x \le -y+1$, and $[x]$ is the integer part of x. This map has two indifferent periodic points $(1,0)$ and $(-1,1)$, and admits an absolutely continuous measure with density $1/(2\log 2)(1-x^2)$. The rate of decay of correlations is proved to have bounds $O(n^{-1+\epsilon})$ $\forall \epsilon > 0$. If we look at the behavior of the map near the periodic orbit, the bounds should be of oeder $O(n^{-1})$.

For invertible systems, we have:

Theorem 4.5 ([H5]). *Let $f : M \to M$ be an almost Anosov diffeomorphism on the surface that satisfies conditions in Theorem 3.9. i). Then f has polynomial decay of correlations with degree $b/2d-1$.*

Proof. (f,μ) can be reduced to an almost expanding map $(\tilde f,\tilde\mu)$ by identify points on each stable curve in every rectangle of a Markov partition. Then we can check that the resulting measure $\tilde\mu$ satisfies the conditions in Theorem 4.2. In Particular, if $x \in P_n$, then $e^{\psi(x)}$ is proportional to $\mu P_{n-1}/\mu P_n = \mu P'_{n-1}/\mu P'_n$ by the definition of ψ. By measuring the length of unstable leaves of P'_n (Proposition 2.6), we know that it is bounded by $\left(\dfrac{n-1}{n}\right)^{\beta}$, $\beta = b/2d$. So we get $\psi(x) \le -\beta/n$, and therefore $(\tilde f,\tilde\mu)$ has decay rate bounded by $O(n^{-(\beta-1)})$. Then we pass the results from $(\tilde f,\tilde\mu)$ to (f,μ). □

4.2. **Convergence to Density Functions.** In this section we study the rate of convergence of $\mathcal{L}_\psi^n g \to \mu(g)$. This is the rate that a system converges to its equilibrium, and the rate that determines the rate of decay of correlations.

Theorem 4.6. *Under the circumstance of Theorem 4.2, for any Lipschitz function g, there is a constant $C > 0$ such that for all $n > 0$,*

$$\int \left|\mathcal{L}_\psi^n g - \mu(g)\right| d\mu \le \frac{C}{n^{\beta-1}};$$

and for any closed subset E with $p \notin E$, there is $C_1 > 0$ such that for all $n > 0$,

$$\left|\mathcal{L}_\psi^n g(x) - \mu(g)\right| \le \frac{C_1}{n^{\beta-1}} \qquad \forall x \in E.$$

Recall the definition of $\mathcal{L}_\psi$. Denote $g_n = \mathcal{L}_\psi^n g$. For $x \in P_n$, $n > 0$, write

$$\mathcal{L}_\psi g(x) = e^{\psi(x_1)} g(x_1) + (1 - e^{\psi(x_1)})\bar g(x_1),$$

where $\bar g(x) = \sum_{\tilde x \in f^{-1}(fx)\cap P_0} e^{\psi(\tilde x)} g(\tilde x) / \sum_{\tilde x \in f^{-1}(fx)\cap P_0} e^{\psi(\tilde x)}$, the average of g with the weight e^ψ at all preimages of fx except for x itself. It is easy to see that

$$\mathcal{L}_\psi^n g(x) = g(x_n)e^{S_n\psi(x_n)} + \sum_{j=1}^{n} \bar g_{n-j}(x_j)\big(1 - e^{\psi(x_j)}\big)e^{S_{j-1}\psi(x_{j-1})}. \tag{4.1}$$

Proof of Theorem 4.6: Note that $\mathcal{O}_{k+1}$ is the complement of Q_k. We proof the following for some large k by induction on n:

$(\mathcal{A}_n)$: $\left|g_n(x) - \mu(g)\right| \le \dfrac{C}{(n+1)^{\beta-1}}$ for any $x \in Q_k$;

$(\mathcal{B}_n)$: $\displaystyle\int_{\mathcal{O}_{k+1}} \left|g_n(x) - \mu(g)\right| d\mu \le \frac{C}{(n+1)^{\beta-1}}$.

"$(\mathcal{A}_j)$, $j = 0, \cdots, n-1$, $\Rightarrow (\mathcal{B}_n)$": Apply (4.1) with the function $g(x) - \mu(g)$. Since $\int_{\mathcal{O}_{k+1}} e^{S_n\psi(x_n)} d\mu = \mu\mathcal{O}_{k+n+1}$ has the order $O(n^{-(\beta-1)})$ by the condition ii') in Theorem 4.2, so is $\int_{\mathcal{O}_{k+1}} e^{S_n\psi(x_n)} \big|g_n(x) - \mu(g)\big| d\mu$.

For the second term, we note that $\int_{\mathcal{O}_{k+1}} \big(1 - e^{\psi(x_j)}\big) e^{S_{j-1}\psi(x_{j-1})} = \mu\mathcal{O}_{k+j} - \mu\mathcal{O}_{k+j+1} = \mu\mathcal{P}_{k+j}$ has the order $O((k+j)^{-\beta})$, and $g_{n-j}(x) - \mu(g)$ has the order $O((n-j)^{-(\beta-1)})$ by the induction hypotheses $(\mathcal{A}_{n-j})$. So the sum is bounded by $C'n^{-(\beta-1)}$ for some small C' if k is large enough.

"$(\mathcal{B}_n) \Rightarrow (\mathcal{A}_n)$": By $(\mathcal{B}_n)$, $\int_{Q_k} \big(g_n(x) - \mu(g)\big) d\mu \le C(n+1)^{-(\beta-1)}$. By Proposition 4.7 below, $g_n(y) - g_n(x)$ is of order $O(n^{-\beta})$ for any $x, y \in Q_k$. So $(\mathcal{A}_n)$ must be true. $\square$

By the proof we can see that the first term in (4.1), and therefore $\mu\mathcal{O}_n$, determines the rate of convergence.

Proposition 4.7. *Let $g > 0$ be a Lipschitz function, and let $k > 0$. Then there is $A > 0$ such that for any $x, y \in Q_k$,*

$$\frac{g_n(y)}{g_n(x)} \le 1 + \frac{A}{(n+1)^\beta} \qquad \forall n > 0.$$

This is the main step towards estimates of the speed of convergence $\mathcal{L}_\psi^n g \to \mu(g)$. We use projective metric (Hilbert metric) to prove the proposition, since under this metric $\mathcal{L}_\psi$ is a contracting operator. Let

$$\mathcal{C}_J(S) = \Big\{ g \in C^0(S) : g(x) > 0 \ \forall x \in S,\ g(y) \le g(x) e^{Jd(x,y)} \ \forall x, y \in S \Big\}$$

be a cone in $C^0(S)$, where $C^0(S)$ is the space of continuous functions on a metric space S. The projective metric of two functions $g, \tilde{g} \in \mathcal{C}_J(S)$ is given by

$$\Delta_J(g, \tilde{g}) = \log \frac{a(g,\tilde{g})}{b(g,\tilde{g})},$$

where

$$a(g,\tilde{g}) = \inf\{a :\ a\tilde{g} - g \in \mathcal{C}_J(S)\}, \quad b(g,\tilde{g}) = \sup\{b :\ g - b\tilde{g} \in \mathcal{C}_J(S)\}.$$

We will always take $\tilde{g} \equiv 1$. So we denote $\Theta_J(g) = \Delta_J(g, 1) = \log(a_g/b_g)$, where

$$a_g = \inf\Big\{a :\ \frac{a - g(y)}{a - g(x)} \le e^{Jd(x,y)}\Big\}, \quad b_g = \sup\Big\{b :\ \frac{g(y) - b}{g(x) - b} \le e^{Jd(x,y)}\Big\}.$$

It is easy to see that to measure how much a function g differs from a constant function, we can use $\Theta_J(g)$, or $g(y) - g(x)$, or $1 - g(y)/g(x)$.

Proof of Proposition 4.7: Suppose k is large. Take l large enough. Recall $Q_k = \bigcup_{i=0}^{n} P_i$. Define $Q_{k,l} = \bigcup_{i=k}^{l} P_i$. We prove the following for all $n \ge 0$:

$(\mathcal{C}_n)$: $\Theta_{2J}(g_n|_{Q_{k+l}}) \le \dfrac{A}{(n+1)^\beta}$;

$(\mathcal{D}_n)$: $\Theta_J(g_n|_{Q_{k,l}}) \le \dfrac{c}{k^{1/\gamma}} \cdot \dfrac{A}{(n+1)^\beta}$;

$(\mathcal{E}_n)$: $\Theta_J(g_n|_{Q_k}) \le \Big(1 - \dfrac{c}{k^{1/\gamma}}\Big) \cdot \dfrac{A}{(n+1)^\beta}$.

Clearly, $(\mathcal{D}_n)$ and $(\mathcal{E}_n)$ imply $(\mathcal{C}_n)$. Since $\mathcal{L}_\psi$ is contracting in the projective metric, we can get that $(\mathcal{C}_n)$ implies $(\mathcal{E}_{n+l})$, if l is large enough. To complete the induction process, we only need the fact that $(\mathcal{C}_j)$, $j = 0, \cdots n-1$, imply $(\mathcal{D}_n)$. It is enough to have $g_n(y) - g_n(x) \le cA(n+1)^{-\beta}$ for some small $c > 0$.

Recall (4.1). For the first term, since $e^{S_n\psi(x_n)}$ has the order $O(n^\beta)$ by the assumption in Theorem4.2 ii'), $e^{S_n\psi(y_n)}g(y_n) - e^{S_n\psi(x_n)}g(x_n)$ has the same order.

For the second term, we note that $d(x_j, y_j)$ has the order $j^{-\alpha}$ for some $\alpha > 1$ since f is expanding in a polynomial rate near the indifferent fixed point. Then we use $(\mathcal{C}_{n-j})$ to obtain that $\bar{g}_{n-j}(y_j) - \bar{g}_{n-j}(x_j) \le C(n-j+1)^{-\beta} d(x_j, y_j) \le C_1(n-j+1)^{-\beta} j^{-\alpha} \le C_2(n+1)^{-\beta}$. Since $\big(1 - e^{\psi(x_j)}\big)e^{S_{j-1}\psi(x_{j-1})}$, $j = 1, \cdots, n$, has the sum $1 - e^{S_n\psi(x_n)}$, it can be treated by the same way as the first term. $\square$

If μ is a σ-finite measure, we cannot expect $\mathcal{L}_\psi^n g \to \mu(g)$ since $\mathcal{L}_\psi^n g(x) \to 0$ for Lebesgue almost every x whenever $|\mu(g)| < \infty$. However, if we multiply the left side by suitable factors, we may still have convergence to the density function.

Theorem 4.8 ([CF])**.** *Suppose f is an almost expanding map on $[0,1]$ that maps both $(0, 1/2)$ and $(1/2, 1)$ to $(0,1)$. Assume that near 0, $fx = x + ax^2 + O(x^3)$. Then for any real function g bounded away from 0 and ∞,*

$$\lim_{n\to\infty} \frac{A_n}{n} \sum_{i=0}^{n-1} \mathcal{L}_\phi^i g = h$$

uniformly on any compact subset $E \subset (0,1]$ for some h with $\mathcal{L}_\phi h = h$, where A_n is of order $\log n$.

By Proposition 4.7, if $0 < \gamma < 1$, then restricted to Q_k, the difference between $\mathcal{L}_\psi^n g$ and a constant is of order $O(n^{-\beta})$. Based on this observation, we make the following conjecture for one dimensional case.

Conjecture 4.9. *Under the circumstance of Theorem 4.3 with $\gamma \ge 1$, for any Lipschitz function g, there exist $a_n = O(n^{-\beta})$, $\beta = 1/\gamma$, such that*

$$\lim_{n\to\infty} A_n \mathcal{L}_\phi^n g = h$$

uniformly on any compact subset $E \subset (0,1]$ for some h with $\mathcal{L}_\phi h = h$, where $A_n = \prod_{i=0}^{n-1}(1 + a_i)$. Further, the convergence rate is of order $n^{-\beta}$.

Since $\{A_n\}$ is convergent if $\gamma \in (0,1)$, the conjecture is consistent with Theorem 4.6.

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