6. Dynamical Systems with Hyperbolic behavior

6.1. **Symbolic dynamical systems.**

6.1.1. *Two-sided full shifts.* Let $Y = \{0, 1, \ldots, K_0 - 1\}$ with the discrete topology. Let $\Sigma = \prod_{n=-\infty}^{\infty} Y$ with the product topology. So any $x \in \Sigma$ has the form

$$x = \{x_n\}_{n=-\infty}^{\infty} = \ldots x_{-2}x_{-1}x_0x_1x_2 \ldots.$$

A neighbourhood basis of a point $\{x_n\}$ consists of the sets

$$E = E_m(x_1 \ldots x_k) := \{y = \{y_n\} \in \Sigma : y_n = x_n \ \forall \ m \leq n \leq m + k\}.$$

which are called *cylinders.* A metric on Σ is given by

$$d(\{x_n\}, \{y_n\} = \sum_{n=-\infty}^{\infty} \frac{\text{sign}\,|x_n - y_n|}{2^{|n|}}.$$

With the metric, Σ is a compact metric space. Σ is called a *symbolic space.*

The *two-sided shift* σ, defined by $\sigma\{x_n\} = \{y_n\}$ with $y_n = x_{n+1}$ is a homeomorphism of Σ. We sometimes write as

$$\sigma(\ldots x_{-2}x_{-1}\overset{*}{x}_0x_1x_2 \ldots) = \ldots x_{-1}x_0\overset{*}{x}_1x_2x_3 \ldots,$$

where the symbol $*$ occurs over the 0th coordinate of each point.

A point $\{x_n\}$ is a fixed point if $x_n = i$ for all $n \in \mathbb{Z}$, where $i \in Y$. $\{x_n\}$ is a periodic point of period $p \in \mathbb{N}$, if and only if $x_n = x_{n+p}$ for all $n \in \mathbb{Z}$.

Proposition 6.1. *Let $\sigma : \Sigma \to \Sigma$ be a two-sided shift.*

(i) *Periodic orbits are dense in Σ, and hence $\Omega(\sigma) = \Sigma$.*
(ii) *σ is topologically mixing, and hence is topologically transitive.*
(iii) *σ is expensive with expensive constant* 1.

6.1.2. *One-sided full shifts.* If we take $\Sigma = \prod_{n=0}^{\infty} Y$, and define *one-sided shift* σ, by $\sigma(\{x_n\}_{n=0}^{\infty}) = \{y_n\}_{n=0}^{\infty}$ with $y_n = x_{n+1}$. That is,

$$\sigma(x_0x_1x_2 \ldots) = x_1x_2x_3 \ldots.$$

σ is a K_0 to one continuous map with

$$\sigma^{-1}(x_0x_1x_2 \ldots) = \{ax_0x_1 \ldots \in \Sigma : \ a \in Y\}.$$

Similar properies as in Proposition 6.1 holds. Also, for any open set $U \subset \Sigma$, there exists $n > 0$ such that $\sigma^n(U) = \Sigma$. In fact, we can find a cylinder $E_m(x_1x_2\dots x_k) \subset U$, then $\sigma^{m+k'}(U) \supseteq \sigma^{m+k'}(E_m(x_1x_2\dots x_k)) \supseteq \Sigma$ for any $k' \geq k$.

6.1.3. *Subshifts of finite type.* Now let $\sigma : \Sigma \to \Sigma$ be the two-sided shift. Let $A = (a_{ij})_{i,j=0}^{k-1}$ be a $k \times k$ matrix with $a_{ij} \in \{0,1\}$ for all $i,j = 0,1,\dots,k-1$. Take $\Sigma_A = \{\{x_n\}_{n=-\infty}^{\infty} \in \Sigma : a_{x_ix_{i-1}} = 1 \forall i \in \mathbb{Z}\}$. In other words Σ_A consists of all the bisequences $\{x_n\}_{n=-\infty}^{\infty}$ whose neighbouring pairs are allowed by the matrix A. The complement of Σ_A is clearly open so Σ_A is a closed and therefore is compact subset of Σ. Also $\sigma(\Sigma_A) = \Sigma_A$ so that $\sigma : \Sigma_A \to \Sigma_A$ is a homeomorphism of Σ_A and is called the *two-sided subshift of finite type* (or *topological Markov' chain*) determined by the matrix A.

If $a_{ij} = 1$ for all $i,j = 0,1,\dots,K_0 - 1$, then $\Sigma_A = \Sigma$, and σ is called *full shift* sometimes. If $A = I$, the identity matrix, then Σ_A consists of exact k fixed points of σ. Sometimcs Σ_A is empty; for example when $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

We call the strings $\underline{w} := w_1w_2\dots w_n$ an *admissible sequence* or *allowable sequence* if $a_{w_\ell,w_{\ell+1}} = 1$ for $\ell = 1,\dots,n-1$. If $n < \infty$ we will also call the string a *word*. We will always assume that the words we use are admissible.

A matrix A is said to be *irreducible* if for any $i,j \in Y$, there exists an $n > 0$ such that the (i,j)th element of A^n is non-zero, or equivalently, there exists an admissible word $\underline{w} := w_1w_2\dots w_n$ such that $w_1 = i$ and $w_n = j$.

A matrix A is said to be *primitive* if there exists an $n > 0$ such that for any $i,j \in Y$, the (i,j)th element of A^n is non-zero, In this case we say that A^n is positive, and write $A^n > 0$. It is easy to see that if $A^n > 0$ for some $n > 0$, then so is A^m for any $m \geq n$.

Clearly a primitive matrix is irreducible. However, the inverse is not true. The matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is irreducible, but not primitive.

Theorem 6.2. *Let $\sigma : \Sigma_A \to \Sigma_A$ be a two-sided subshift of finite type. Then*

(i) *σ is topologically transtive if A is irreducible,*
(ii) *σ is topologically mixing if A is primitive.*

Proof. (i) Take open sets $U, V \subset \Sigma_A$. We may assume that there are cylinders given by words $\underline{u} = u_{-r}\dots u_r$ and $\underline{v} = v_{-r}\dots v_r$ of length $2r+1$ such that $E_{-r}(\underline{u}) \subset U$ and $E_{-r}(\underline{v}) \subset V$.

Since A is irreducible, there exists a word $\underline{w}$ of length $n > 0$ such that $v_r\underline{w}u_{-r}$ is an admissible word. Let $t = 2r+1+n$. Since $E_{-r}(\underline{vwu}) \subset$

$E_{-r}(\underline{v})$ and $\sigma^t E_{-r}(\underline{vwu}) = E_{-r-t}(\underline{vwu}) \subset E_{-r}(\underline{u})$, we get that that

$$E_{-r}(\underline{vwu}) \subset \sigma^{-t} E_{-r}(\underline{u}) \cap E_{-r}(\underline{v}) \subset \sigma^{-t} U \cap V.$$

By Theorem 5.1 σ is topologically transitive.

(ii) The proof is similar. □

Lemma 6.3. *For every $i, j \in \{0, 1, \ldots, K_0 - 1\}$, the number N_{ij}^m of admissible words of length $m+1$ that begin at i and end at j is equal to the (i, j) entry a_{ij}^m of the matrix A^m.*

Proof. We use induction on m. First, it follows from the definition of A that $N_{ij}^1 = a_{ij}$.

We now assume for some $m \in \mathbb{N}$, $N_{ij}^m = a_{ij}^m$ for all ij. Note that by matrix multiplication

$$a_{ij}^{m+1} = \sum_{k=0}^{K_0-1} a_{ik}^m a_{kj} = \sum_{k=0}^{K_0-1} N_{ik}^m a_{kj}.$$

An admissible words of length $m+2$ connecting i to j is an admissible words of length $m+1$ connecting i to some k followed by j with $a_{kj} = 1$. So the right side of the formula also gives N_{ik}^{m+1}. Hence the result follows by induction. □

Let $P_n(T)$ denote the set of periodic orbits of T of period n, that is, the set of fixed point of T^n, and $\operatorname{Card} P_n(T)$ denote cardinality of $P_n(T)$.

Corollary 6.4. $\operatorname{Card} P_n(\sigma_A) = \operatorname{tr} A^n$.

Proof. If $x = \{x_n\} \in P_n(\sigma_A)$, then $x_0 = x_n$. So each periodic orbit corresponding a word of length $n+1$ that begins and ends at the same element $i \in \{0, 1, \ldots, K_0 - 1\}$. There are a_{ii}^n such words. □

Consider a two sided shift $\sigma : \Sigma \to \Sigma$. For any $x = \{x_n\}_{n=-\infty}^{\infty} \in \Sigma$, Denote

$$\begin{aligned} W^s(x) =& \big\{ y = \{y_n\}_{n=-\infty}^{\infty} \in \Sigma : \ \exists N \in \mathbb{Z} \text{ s.t. } x_n = y_n \ \forall n \geq N \big\} \\ W^u(x) =& \big\{ y = \{y_n\}_{n=-\infty}^{\infty} \in \Sigma : \ \exists N \in \mathbb{Z} \text{ s.t. } x_n = y_n \ \forall n \leq N \big\} \end{aligned}$$

Then it is easy to see that

$$\begin{aligned} \lim_{n\to\infty} d(\sigma^n x, \sigma^n y) = 0 \qquad & \forall y \in W^s(x) \\ \lim_{n\to\infty} d(\sigma^{-n} x, \sigma^{-n} y) = 0 \qquad & \forall y \in W^u(x) \end{aligned}$$

The sets $W^s(x)$ and $W^u(x)$ are called the *stable set* and *unstable set* of x respctively.

6.2. Expanding systems.

Definition 6.5. *Let X be a compact metric space, and $T : X \to X$ is continuous. T is said to be* uniformly expanding *or simply* expanding *if there exists $\kappa < 1$ such that*

$$d(Tx, Ty) \geq \kappa^{-1}\rho d(x, y) \qquad \forall x, y \in X \quad \textit{with} \quad d(x, y) \leq \epsilon_0,$$

where $\epsilon_0 > 0$.

We say that $T : X \to X$ is *piecewise continuous* if there is a finite or countable partition X into $\{X_n\}$ such that for each n, $T|_{X_n} : X_n \to X$ is continuous.

Expansion can be generalized to piecewise continuous maps if we replace the requirement $x, y \in X$ by $x, y \in X_n$ for each n.

Clearly, expansion implies expansiveness.

Example 6.6. *Let $X = I = [0, 1]$ and $T(x) = 2x \pmod 1$. T is a piecewise continuous expanding map. If we identify 0 and 1, then X becomes a circle, and T is continuous on the circle.*

Take a partition $X = I_0 \cup I_1$, where $I_0 = [0, 1/2)$ and $I_1 = [1/2, 1)$. We have $T(I_i) = I$ for both $i = 0, 1$.

Let $Y = \{0, 1\}$ and define $\Sigma^+ = \prod_{i=0}^{\infty}$. Define $\pi : \Sigma^+ \to I$ such that for any $x = \{x_0x_1x_2 \ldots\}$, $x_n = 0$ or 1, $\pi(x) = t \in [0, 1]$ is the element such that $T^n(t) \in I_{x_n}$. In fact, t is the number whose binary expansion is $0.x_0x_1x_2\ldots$. It is easy to check t is unique by expansiveness. In fact, if $s, t \in I$ such that $T^n(t), T^n(s) \in I_{x_n}$ for all $n > 0$, then we have $d(T^n(t), T^n(s)) = 2^n d(s, t) \to \infty$, a contradiction. So π is well defined.

It is also easy to see that π is one to one, except $\pi(0^\infty) = \pi(1^\infty)$ and $\pi(w10^\infty) = \pi(w01^\infty)$ for any word w. The images of 0 and the numbers of the form $k/2^n$.

Clearly we have $\pi\sigma = T\pi$. So π gives a toplogical semiconjugacy, and it is in fact a conjugacy in a G_δ set. So we can get that T is topologially transitive and mixing, and has dense periodic orbits.

In general, suppose $T : X \to X$ is a map that has a Markov partition, that is, there exists a collection $\{R_0, R_1, \ldots, R_{K_0-1}\}$ of subsets X with $\overline{\operatorname{int} R_i} = R_i$, $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ for $i \neq j$, and $X = \cup_{i=0}^{K_0-1} R_i$ such that $T(R_i) = X$. Then for any $x = \{x_n\}_{n=0}^{\infty}$, let $\pi(x) = \cap_{n=0}^{\infty} T^{-n} R_{x_n}$. If T is expanding, then $\operatorname{diam}\big(\cap_{i=0}^{n} T^{-i} R_{x_i}\big) \to 0$ as $n \to \infty$. Hence $\pi(x)$

contains exact one point in X. We can check that

$$\begin{aligned} T\pi(x) =& T\bigcap_{n=0}^{\infty} T^{-n}R_{x_n} = \bigcap_{n=0}^{\infty} T^{-n+1}R_{x_n} \\ =& \bigcap_{n=0}^{\infty} T^{-n}R_{x_{n+1}} = \pi\big(\{x_{n+1}\}_{n=0}^{\infty}\big) = \pi\sigma(x). \end{aligned}$$

Example 6.7. *Let $I = [0,1]$ and define $f : I \to I$ by*

$$T(x) = \begin{cases} 3x & x \in [0,1/3]; \\ 3x-1 & x \in (1/3,2/3); \\ 3x-2 & x \in [2/3,1]. \end{cases}$$

Then let $X = \{x \in I : T^n(x) \notin (1/3,2/3), n \geq 0\}$. X is a standard Cantor set that is invariant under T.

It is easy to see that (X,T) is topologically conjugate to a symbolic system (Σ,σ) of two symbols.

Example 6.8 (Quadratic maps)**.** *Let $X = I = [0,1]$ and let $T(x) = 4\lambda x(1-x)$, where $0 \leq \lambda \leq 4$.*

This is NOT an expanding map for any λ since $T'(1/2) = 0$.

This map has complicated dynamics for large values of the parameter. For smaller values of the parameter, the following can be proved

Lemma 6.9. *For $0 \leq \lambda \leq 1$ all orbits of $f_\lambda(x) = \lambda x(1-x)$ on $[0,1]$ are asymptotic to 0.*

For $1 < \lambda < 3$ all orbits of $T_\lambda(x) = 4\lambda x(1-x)$ on $[0,1]$, except for 0 and 1, are asymptotic to the fixed point $x_\lambda = 1-(1/\lambda)$.

For $\lambda = 4$, the map is topologically conjugate to the map $S : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $S(t) = 2t$.

Lemma 6.10. *The map $T : [-1,1] \to [-1,1]$, $T(x) = 2x^2 - 1$ is topologically conjugate to S.*

Proof. Let $h(t) = \cos t$. Then

$$Th(t) = T(\cos t) = 2\cos^2 t - 1 = \cos 2t = hS(t). \qquad \square$$

Example 6.11. *Let A be an $n \times n$ matrix with all eigeinvalues greater than 1. Then the map $x \to Ax$ is expanding from $\mathbb{R}^n$ to itself. In particular, if A is an integral matrix, then A induce a expanding map $Tx = Ax \pmod{\mathbb{Z}^n}$ on the torus $\mathbb{T}^n$.*

It can be proved that such map is topologically conjugate to an one sided shift.

In fact, all the expanding maps on the torus is topologically conjugate to an one sided shift.

6.3. **Hyperbolic systems.**

Definition 6.1. *Suppose M is a compact Riemannian manifold, and f is a diffeomorphism defined on M. A closed invariant set Λ, i.e., $f(\Lambda) = \Lambda$, is said to be a* hyperbolic set *if at every point $x \in \Lambda$ we have an invariant decomposition*

$$T_xM = E^u_x \oplus E^s_x, \quad Df_xE^u_x = E^u_{fx},\ Df_xE^s_x = E^s_{fx},$$

and constants $C > 0$ and $\kappa < 1$ such that for all $n \in \mathbb{N}$

$$\|Df^{-n}(x)|_{E^u_x}\| \le C\kappa^n, \qquad \|Df^n(x)|_{E^s_x}\| \le C\kappa^n.$$

By choose a suitable Riemannian metric, we can make $C = 1$.

Example 6.12 (Baker's transformation)**.** *Let $f : I^2 \to I^2$ such that*

$$f(x,y) = \Big(2x - \lfloor 2x \rfloor,\ \frac{y + \lfloor 2y \rfloor}{2}\Big).$$

This is a piecewise differentiable map.

By the same analysis we can see that the map is topologically semi-conjugate to a two sided full shift ot two symbols 0 and 1. 0 and 1. In fact, we can take $R_0 = [0, 1/2] \times [0, 1]$ and $R_1 = [1/2, 1] \times [0, 1]$. Then for any $x = \{x_n\}_{n=-\infty}^{\infty} \in \Sigma$, $x_n \in \{0, 1\}$, we an define

$$\pi(x) = \cap_{n=-\infty}^{\infty} f^{-n} R_{x_n},$$

which is the conjugacy map $\pi : \Sigma \to I^2$.

For any $x = (x^{(1)}, x^{(2)}) \in I^1$, The sets $W^s(x) = \{(t, x^{(2)}) \in I^2 : t \in I\}$ and $W^u(x) = \{(x^{(1)}, t) \in I^2 : t \in I\}$ are called the *stable* and *unstable manifold* at x respectively.

Example 6.13 (Horseshoe map)**.** *Let $\Delta \subset \mathbb{R}^2$ be a square. Let $f \colon \Delta \to \mathbb{R}^2$ a diffeomorphism of Δ onto its image such that the intersection $\Delta \cap f(\Delta)$ consists of two "horizontal" rectangles Δ_0 and Δ_1 and the restriction of f to the components $\Delta^i := f^{-1}(\Delta_i)$, $i = 0, 1$, of $f^{-1}(\Delta)$ is a hyperbolic linear map, contracting in the vertical direction and expanding in the horizontal direction. This implies that the sets Δ^0 and Δ^1 are "vertical" rectangles. One of the simplest ways to achieve this effect is to bend Δ into a "horseshoe", or rather into the shape of a permanent magnet, although this method produces some inconveniences with orientation.*

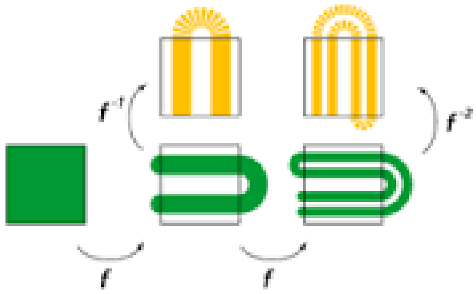

FIGURE 1. Horseshoe

The horeshoe map was introduced by Stephen Smale while studying the behavior of the orbits of the van der Pol oscillator. The map is regarded as a hallmark of chaos.

Example 6.14 (Solenoid). *Consider the solid torus $M := \mathbb{S}^1 \times \mathbb{D}^2$, where $\mathbb{D}^2$ is the unit disk in $\mathbb{R}^2$. This looks like a bagel. On it we define coordinates (φ, x, y) such that $\varphi \in S^1$ and $(x, y) \in \mathbb{D}^2$, i.e., $x^2+y^2 \leq 1$. Using these coordinates we define the map by doubling up and shrinking the thickness by five.*

$$f(\varphi, x, y) = \left(2\varphi, \quad \frac{1}{5}x + \frac{1}{2}\cos\varphi, \quad \frac{1}{5}y + \frac{1}{2}\sin\varphi\right)$$

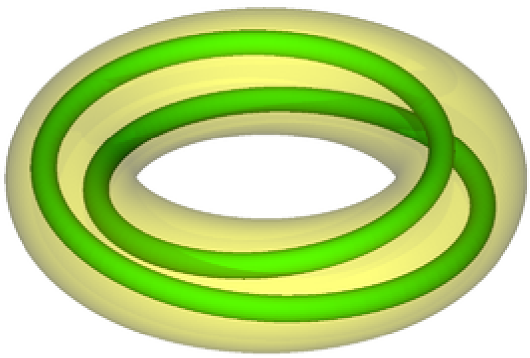

FIGURE 2. Solenoid

The solenoid S is the set of points that stay inside $\mathbb{D}^2$ under f^n for all $n \in \mathbb{Z}$. That is, $S = \cap_{n\geq 0} f^n M$.

Clearly, $f : S \to S$ is topologically conjugate to a full shift.

Definition 6.2. *A closed subset Λ of M is said to be an* attractor *if there is a neighborhood U of Λ such that*

$$\Lambda = \cap_{n\geq 0} f^n U.$$

A solenoid is a *hyperbolic attractor*. Because of the chaotic behavior it is a *stange attractor*.

Example 6.15 (Hyperbolic toral automorphisms)**.** *Let* $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. *Since* $\det A = 1$, *A induces an automorphism on the torus* $\mathbb{T}^2$ *given by*

$$f(z) = Az \quad (\mod \mathbb{Z}^2) \qquad \forall z \in \mathbb{T}^2$$

or

$$f(x, y) = (2x + y,\ x + y) \quad (\mod \mathbb{Z}^2) \qquad \forall (x, y) \in \mathbb{T}^2.$$

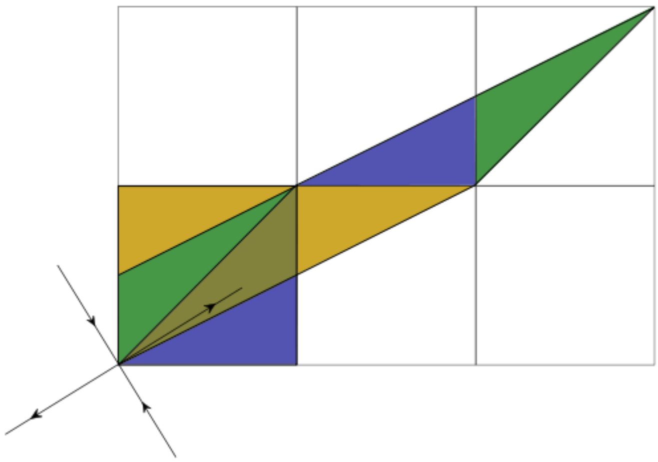

FIGURE 3. The hyperbolic toral automorphism

A has two eigenvalues

$$\lambda_1 = \frac{3 + \sqrt{5}}{2} > 1, \quad \lambda_2 = \frac{3 - \sqrt{5}}{2} < 1.$$

The eigenvectors corresponding to the eigenvalues are given by

$$y = \frac{\sqrt{5} - 1}{2} x, \qquad y = -\frac{\sqrt{5} + 1}{2} x.$$

They are the unstable space E^u_x and stable space E^s_x at each tangent space $T_x\mathbb{T}^2$.

For any $x \in \mathbb{T}^2$, the stable and unstable manifolds at x are the sets $W^s(x) = \{x + tv^s \ (\mod \mathbb{Z}^2) : t \in \mathbb{R}\}$ and $W^u(x) = \{x + tv^u \ (\mod \mathbb{Z}^2) : t \in \mathbb{R}\}$ respectively, where $v^s \in E^s_x$ and $v^u \in E^u_x$.

Proposition 6.16. *Periodic points of f are dense and* $\operatorname{Card} P_n(f) = \lambda_1^n + \lambda_2^n - 2$.

Proof. It can be proved that a point is periodic if and only if the coordinates of the point are rational.

Now we calculate $\operatorname{Card} P_n(f)$. The map

$$G = f^n - \mathrm{id} \colon x \to A^n x - x = (A^n - I)x \pmod{\mathbb{Z}^2}$$

is a well-defined noninvertible map of the torus onto itself. Note that if $f^n(x) = x$, then $f^n(x) - x = 0$ on $\mathbb{T}^2$, and hence $G(x) = 0 \pmod{\mathbb{Z}^2}$, that is, the set $P_n(f)$ of fixed points of f^n are exactly the preimages of the point $0 \in \mathbb{T}^2$ under G: $P_n(f) = G^{-1}(0)$. (Actually, every point $z \in \mathbb{T}^2$ has the same number of preimages.) Lift to $\mathbb{R}^2$, we have the map $\overline{G} : [0,1) \times [0,1) \to \mathbb{R}^2$ given by $\overline{G}(x) = (A^n - I)x$. Then we get $\overline{G}(P_n(f)) = \overline{G}(G^{-1}(0)) \in \mathbb{Z}^2$. That is, the number of elements in $P_n(f)$ is exactly equal to the number of the points of $\mathbb{Z}^2$ in $\overline{G}([0,1) \times [0,1))$, which is equal to the area of the image: $|\det(A^n - \mathrm{id})| = |(\lambda_1^n - 1)(\lambda_2^n - 1)| = \lambda_1^n + \lambda_2^n - 2$. $\square$

Remark. *The area of a parallelogram with integer vertices is the number of lattice points it contains, where points on the edges are counted as half, and all vertices count as a single point.*

The toral automorphism can be coding by a subshift of finite type by using the partition first into $\{R^{(1)}, R^{(2)}\}$, and then a finer partition into $\{\Delta_0, \Delta_1, \Delta_2, \Delta_3, \Delta_4\}$ with $R^{(1)} = \Delta_0 \cup \Delta_1 \cup \Delta_2$ and $R^{(2)} = \Delta_3 \cup \Delta_4$.

From Figure 5 we can see that $f(\Delta_0)$, $f(\Delta_1)$ and $f(\Delta_2)$ cross Δ_0, Δ_1 and Δ_3 in E^u direction, while $f(\Delta_3)$ and $f(\Delta_4)$ cross Δ_2 and Δ_4 in E^u direction. So we can get a 5×5 matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

It follows that $\sigma : \Sigma_A \to \Sigma_A$ is topologically semiconjugate to f. Since $A^2 > 0$, f is topologically mixing. It is clear that f is expansive with expansive constant greater than $\lambda_2/2$.

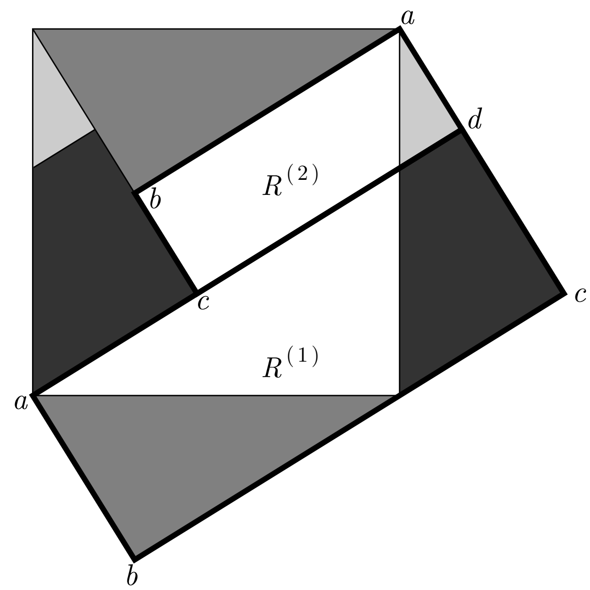

FIGURE 4. Partitioning the torus.

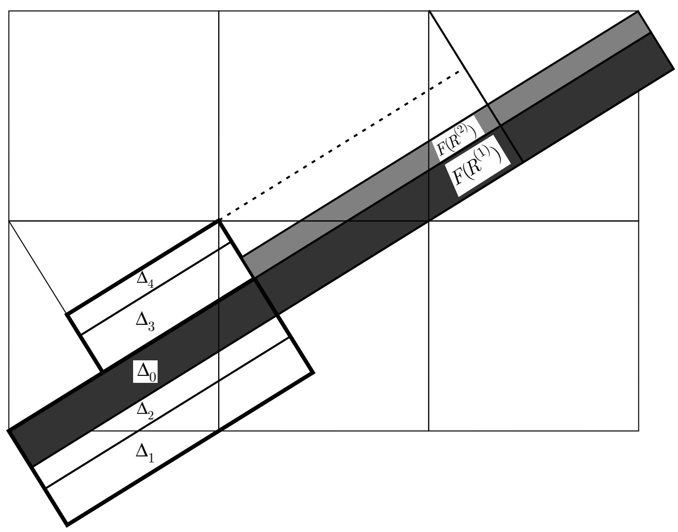

FIGURE 5. The image of the partition.