# 5. Topological Properties

## 5.1. **Topological transitivity.** Let $X$ be a compact topological space.

**Definition 5.1.** *A homeomorphism $T : X \to X$ is called* topologically transitive *if there is some $x \in X$ such that $O(x)$ is dense in $X$.*

**Remark.** *If $T : X \to X$ is continuous,* topological transitivity *is defined as $O_+(x)$ is dense in $X$ for some $x \in X$. Sometimes it is also called* one-sided topological transitivity

A set which is the intersection of a countable collection of open sets is called a $G_\delta$.

**Theorem 5.1.** *The following are equivalent for a homeomorphism $T : X \to X$ of a compact topological space.*

(i) *$T$ is topologically transitive.*

(ii) *Whenever $E$ is a closed subset of $X$ and $TE = E$ then either $E = X$ ,or $E$ is nowheere dense (or, equivalently, whenever $U$ is an open subset of $X$ with $TU = U$ then $U = \emptyset$ or $U$ is dense).*

(iii) *Whenever $U, V$ are non-empty open sets then there exists $n \in \mathbb{Z}$ with*

$$T^n(U) \cap V \neq \emptyset.$$

(iv) *$\{x \in X : \overline{O(x)} = X\}$ is a dense $G_\delta$.*

*Proof.* (i) $\Rightarrow$ (ii) Suppose $\overline{O(x_0)} = X$ and let $E \neq \emptyset$ be a closed subset with $TE = E$.

If there is an open set $U \subset E$, $U \neq \emptyset$, then there exists $p \in \mathbb{Z}$ such that $T^p(x_0) \in U \subset E$, so that $O(x_0) \subset E$ and therefore $X \subset E$. We get $E = X$.

If otherwise there is not open set $U \subset E$, then $E$ is a nowhere dense set.

(ii) $\Rightarrow$ (iii). Suppose $U, V \neq \emptyset$ are open sets. Then $\cup_{n=-\infty}^{\infty} T^n U$ is a $T$-invariant open set, so it is necessarily dense by condition (ii), Thus $\cup_{n=-\infty}^{\infty} T^n U \cap V \neq \emptyset$.

(iii) $\Rightarrow$ (iv). Let $U_1, U_2, \ldots, U_n, \ldots,$ be a countable base for $X$. It is easy to verify $\{x \in X : \overline{O(x)} = X\} = \cap_{n=1}^{\infty} \cup_{m=-\infty}^{\infty} T^m U_n$ is clearly dense by condition (iii). Hence the result follows.

(iv) $\Rightarrow$ (i). This is clear. $\square$

The noninvertibale version of the theorem is the following.

**Theorem 5.2.** *The following are equivalent for a continuous map $T : X \to X$ of a compact topological space with $TX = X$.*

(i) *$T$ is one sided topologically transitive.*

(ii) *Whenever $E$ is a closed subset of $X$ and $E \subset T^{-1}E$ then either $E = X$ ,or $E$ is nowheere dense (or, equivalently, whenever $U$ is an open subset of $X$ with $T^{-1}U \subset U$ then $U = \emptyset$ or $U$ is dense).*

(iii) *Whenever $U, V$ are nonempty open sets then there exists $n \in \mathbb{Z}$ with*
$$T^{-n}(U) \cap V \neq \emptyset.$$

(iv) $\{x \in X : \overline{O_+(x)} = X\}$ *is a dense $G_\delta$.*

**Remark.** *The difference between the theorems is that $TE = E$, $TU = U$, $T^n(U) \cap V \neq \emptyset$, and $\{x \in X : \overline{O(x)} = X\}$ are replaced by $E \subset T^{-1}E$, $T^{-1}U \subset U$, $T^{-n}(U) \cap V \neq \emptyset$, and $\{x \in X : \overline{O_+(x)} = X\}$ respectively.*

**Theorem 5.3.** *Let $T : X \to X$ be a homeomorphism. Then $T$ is one-sided topologically transitive iff $T$ is topologically transitive and $\Omega(T) = X$.*

*Proof.* "$\Rightarrow$" Suppose $\{T^n(x_0) : n \geq 0\}$ is dense in $X$. Clearly $T$ is topologically transitive.

Suppose $\Omega(T) \neq X$. Then there is a nonempty open set $U$ such that $\{T^n U : n \geq 0\}$ are pairwise disjoint sets. Hence $\{T^n U : n \in \mathbb{Z}\}$ are pairwise disjoint sets. On the other hand, there exists $n_0 > 0$ such that $T^{n_0}(x_0) \in U$. Hence, $T^{n+n_0}(x_0) \in f^n U$ for any $n > 0$. So only $\{x_0, \ldots, f^{n_0-1}(x)\}$ can belong to $\cup_{i=1}^{\infty} T^{-i}U$. Since $\{T^{-i}U : i > 0$ are pairwise disjoint, $\{T^n(x_0) : n \geq 0\}$ does not intersect some $T^{-i}U$, contradictiong topological transitivity.

"$\Leftarrow$" Now suppose $T$ is topologically transitive and $\Omega(T) \neq X$. Let $U, V$ be nonempty open sets. By (iii) of Theorem 5.1 we know there is some $N \in \mathbb{Z}$ with $W := T^N U \cap V \neq \emptyset$ so we may as well suppose $N \geq 0$. Since $\Omega(T) \neq X$, there exists $n \geq N + 1$ with $T^{-n}W \cap W \neq \emptyset$. Then $T^{-(n-N)}U \cap V \supset T^{-n}W \cap W \neq \emptyset$. By (iii) of Theorem 5.2, we get that $T$ is one-sided topologically transitive, $\square$

**Example 5.4.** *Let $X = \{\exp(2i \tan^{-1} n) : n \in \mathbb{Z}\} \cup \{\exp(\pi i)\} \subset \mathbb{S}^1$. Clearly $X$ is a compact metric space with a limit point $\exp(\pi i)$. Define $T : X \to X$ by $T(\exp(\pi i)) = \exp(\pi i)$, and $T(\exp(2i \tan^{-1} n)) = \exp(2i \tan^{-1}(n+1))$ for $n \in \mathbb{Z}$. Then $T$ is topologically transitive, but is not one-sided topologically transitive.*

A function $f$ is invariant if $f(Tx) = f(x)$ for every $x \in X$.

**Theorem 5.5.** *If $T$ is a topologically transitive homeomorphism or a one- sided topologically transitive continuous map then $T$ has no non-constant invariant cominuous function.*

*Proof.* If $f \circ T = f$, then $f \circ T^n = f$. So $f$ is constant on orbits of points. The rcsult then follows. $\square$

## 5.2. **Topological mixing.**

**Definition 5.6.** *A continuous map $T : X \to X$ is topologically mixing if for any nonempty open sets $U$ and $V$, there exists $N \in \mathbb{Z}$ such that for any $n > N$,*

$$T^{-n}(U) \cap V \neq \emptyset.$$

Recall that by part (iii) in Theorem 5.1 or 5.2, $T : X \to X$ is topologically transitive iff there exists $n \in \mathbb{N}$ such that $T^{-n}(U) \cap V \neq \emptyset$. So topological mixing implies topological transitivity. But the inverse is not true. For example, an irrational circle rotation is topologically transitive but not topologically mixing.

## 5.3. **Expansiveness.**

In the next definition we require that $X$ is a metric space.

**Definition 5.7.** *A homeomorphism $T$ of a compact metric space $X$ is said to be* expansive *if $\exists \delta > 0$ with the property that if $x \neq y$ then $\exists n \in \mathbb{Z}$ with $d(T^n x, T^n y) > \delta$. We call $\delta$ an* expansive constant *for $T$.*

**Remark.** *A continuous map $T$ on $X$ is said to be* positively expansive *if $\exists \delta > 0$ with the property that if $x \neq y$ then $\exists n \in \mathbb{N}$ with $d(T^n x, T^n y) > \delta$.*

For a topological space $X$, an *open cover* is a collection of open sets $\alpha = \{A_i\}_{i \in I}$, where $I$ is an index set, such that $X = \cup_{i \in I} A_i$.

If $\alpha = \{A_i\}_{i \in I}$ and $\beta = \{B_j\}_{j \in J}$ are covers of $X$, denote by $\alpha \vee \beta$ the open cover whose elemtnts have the form $\{A \cap B : A \in \alpha, B \in \beta\}$. If $T : X \to X$ be a homeomorphism, denote by $T^{-1}\alpha$ the cover whose elemtnts has the form $\{T^{-1}A : A \in \alpha\}$.

For a metric space $X$, if $\alpha$ is a finite open cover, then there exists number $\delta > 0$ such that each subset of $X$ of diameter less than or equal to $\delta$ lies ill some member of $\alpha$. Such a number $\delta > 0$ is called a *Lebesgue number* for $\alpha$.

**Definition 5.8.** *Let $X$ be a compact topological space and $T : X \to X$ a homeomorphism. A finite open cover $\alpha$ of $X$ is a* generator *for $T$ if for every bisequence $\{A_n\}_{n \in \mathbb{Z}}$ of members of $\alpha$ the set $\cap_{n \in \mathbb{Z}} T^{-n} \bar{A}_n$ contains at most one point of $X$.*

**Theorem 5.9.** *Let $T$ be a homeomorphism of a compact metric space $X$. Then $T$ is expansive iff $T$ has a generator.*

*Proof.* "$\Rightarrow$" Let $\delta$ be an expansive constant for $T$. Take any finite cover $\alpha$ consisting of open balls of radius $\delta/2$. Suppose $x, y \in \cap_{n\in\mathbb{Z}} T^{-n}\bar{A}_n$, where $A_n \in \alpha$. Then $d(T^n x, T^n y) \leq \delta$ for all $n \in \mathbb{Z}$ so $x = y$ by expansiveness. Therefore $\alpha$ is a generator.

"$\Rightarrow$" Conversely, suppose $\alpha$ a is a generator. Let $\delta$ be a Lebesgue number for $\alpha$. If $d(T^n x, T^n y) \leq \delta$ for all $n \in \mathbb{Z}$, then $\forall n \in \mathbb{Z}$, there exists $A_n \in \alpha$ with $T^n x, T^n y \in A_n$ and so,

$$x, y \in \cap_{n\in\mathbb{Z}} T^{-n} A_n$$

Since this intersection contains at most one point we have $x = y$. Hence $T$ is expansive. $\square$

**Corollary 5.10.** (i) *Expansiveness is independent of the mefric as long as the metric gives the topology of $X$. (However the expansive constant does change.)*

(ii) *If $k \neq 0$, then $T$ is expansive iff $T^k$ is expansive.*

(ii) *Expansiveness is topological conjugacy invariant, i.e. if, for $i = 1, 2$, $T_i : X_i \to X_i$ is a homeomorphism of a compact metric space and if $\phi : X_1 \to X_2$ is a homeomorphism with $\phi T_1 = T_2\phi$, then $T_1$ is expansive iff $T_2$ is expansive.*

*Proof.* (i) This is because the concept of generator does not depend on the metric.

(ii) If $\alpha$ is a generator for $T$, then $\alpha \vee T^{-1}\alpha \vee \ldots \vee T^{-(k-1)}\alpha$ is a generator for $T^k$. Also any generator for $T^k$ is a generator for $T$.

(iii) A cover $\alpha$ is a generator for $T_2$ iff $\phi^{-1}\alpha$ is a generator for $T_1$ $\square$

**Remark.** *Note that toplogical transitivity for $T$ does not imply toplogical transitivity of $T^k$.*