4. Circle Homeomorphisms

4.1. **Rotation numbers.** Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation preserving homeomorphism. Let $\pi : \mathbb{R} \to \mathbb{S}^1$ be the map $\pi(t) = \exp(2\pi i t)$.

Lemma 4.1. *There is a continuous map $F : \mathbb{R} \to \mathbb{R}$ such that*

(i) $\pi F = f\pi$*;*
(ii) F *is monotone increasing;*
(ii) $F - \mathrm{id}$ *is periodic with period* 1*.*

Moreover, any two such maps differ by an integer translation.

Proof. Define $F(0)$ to be any number in the set $\pi^{-1}f(\pi(0))$. Let U and V be neighborhoods of 0 and $F(0)$ respectively that have length less than 1. Note that $\pi|_V : V \to \pi(V)$ is a homeomorphism. For any $t \in U$, define $F(t) = (\pi|_V)^{-1} \circ f(\pi(t))$ whenever it is defined. Then F is extended to a neighborhoods $U' \subseteq U$. Using the same way we can extend the definition of F to $\mathbb{R}$. It is easy to check (i)-(iii).

Suppose $G : \mathbb{R} \to \mathbb{R}$ is also a such map. Then by (i) we have that for any $t \in \mathbb{R}$, $\pi(G(t)) = f(\pi(t)) = \pi(F(t))$. That is, there exists an integer $n = n_t$ such that $G(t) = F(t) + n_t$. Since both F and G are continuous, and n_t must be a integer, it must be independent of t. □

Note that (i) implies that F is a homeomorphism. We call such an F a *lift* of f.

Proposition 4.2. *Given F as above, the limit*

$$\tau(F) := \lim_{n\to\infty} \frac{F^n(x)}{n}$$

exists for each $x \in \mathbb{R}$, and and is independent of x.

Proof. (1) Independence of x:

Since $F(x+1) = F(x)+1$ for all x, it follows that $F^n(x+1) = F^n(x)+1$ for all x and n. Now, suppose that $x \le y \le x+1 \le y+1$. Since F^n is monotone increasing, using $F^n(x+1) = F^n(x)+1$, we have

$$\frac{F^n(x)}{n} \le \frac{F^n(y)}{n} \le \frac{F^n(x+1)}{n} \le \frac{F^n(y+1)}{n}.$$

This implies that if the limit $\lim_{n\to\infty} \frac{F^n(x)}{n}$ exists, then so does $\lim_{n\to\infty} \frac{F^n(y)}{n}$, and they are equal.

(2) Existence if f has a periodic point:

Let x be a periodic point of period m, and let $y \in \mathbb{R}$ be such that $\pi(y) = x$. Then there is an integer p such that $F^m(y) = y + p$. Then, $F^{nm}(y) = y + np$. So

$$\lim_{n\to\infty} \frac{F^{nm}(x)}{nm} = \lim_{n\to\infty} \frac{y+np}{nm} = \frac{p}{m}$$

Now, for any integer k, let $k = rm + q$ with $0 \le q < m$. Then,

$$\frac{F^k(y)}{k} = \frac{F^k(y) - F^{rm}(y) + F^{rm}(y)}{k}$$

and

$$\left|\frac{F^k(y) - F^{rm}(y)}{k}\right| \leq \frac{M}{k}$$

where $M = \max_{0 \leq q < m} \left|F^q(y) - y\right|$. Thus,

$$\lim_{k\to\infty} \frac{F^k(x)}{k} = \lim_{k\to\infty} \frac{F^{rm}(x)}{k} = \lim_{r\to\infty} \frac{F^{rm}(y)}{rm} = \frac{p}{m}.$$

Thus, the limit exists if f has a periodic point.

(3) Existence if f has no periodic points:

This implies that $F^m(x) - x$ is not an integer for any $m > 0$ and any $x \in \mathbb{R}$. Let p_m be an integer such that

$$p_m < F^m(0) < p_m + 1 \tag{4.1}$$

Therefore, for all $x \in \mathbb{R}$, $p_m < F^m(x) - x < p_m + 1$, since if otherwise, then by the Intermediate Value Theorem, we have $F^m(y) - y = p_m$ or $F^m(y) - y = p_m + 1$ for some y, which is a contradiction. Hence, for $1 \leq i \leq n$, $p_m < F^{im}(0) - F^{(i-1)m}(0) < p_m + 1$. Adding together these inequalities for $i = 1, \ldots, n$, the middle terms telescope, and we get

$$np_m < F^{nm}(0) < n(p_m + 1) \tag{4.2}$$

Dividing (4.1) by m and (4.1) by mn, we get that $\dfrac{F^{nm}(0)}{mn}$ and $\dfrac{F^m(0)}{m}$ are both in the interval $\left(\dfrac{p_m}{m}, \dfrac{p_m+1}{m}\right)$. So

$$\left|\frac{F^{nm}(0)}{mn} - \frac{F^m(0)}{m}\right| \leq \frac{1}{m}.$$

Interchanging the roles of m and n, we get

$$\Big|\frac{F^{nm}(0)}{mn} - \frac{F^n(0)}{n}\Big| \leq \frac{1}{n},$$

and, hence,

$$\Big|\frac{F^m(0)}{m} - \frac{F^n(0)}{n}\Big| \leq \frac{1}{m} + \frac{1}{n}.$$

Hence, the sequence $\left\{\frac{F^n(0)}{n}\right\}$ is a Cauchy sequence, and thus has a limit. □

Lemma 4.3. *Let F and G are both lift of f, then there exists $p \in \mathbb{Z}$ such that $\tau(G) = \tau(F) + p$.*

Proof. Since F and G are both lift of f, then there exists $p \in \mathbb{Z}$ such that $G(x) = F(x) + p$ for any $x \in \mathbb{R}$. So we have $G^2(x) = G(G(x)) = F(F(x) + p) + p = F(x) + 2p$, and for each $n > 0$, $G^n(x) = F^n(x) + np$. Hence,

$$\tau(G) = \lim_{n\to\infty} \frac{G^n(x)}{n} = \lim_{n\to\infty} \frac{F^n(x) + np}{n} = \tau(F) + p.$$

□

The above lemma says that $\tau(f)$ is independent of the choice of the lift F.

Definition 4.1. *The number $\tau(f) := \pi\tau(F)$ is called the* rotation number *of f.*

We say that $\tau(f)$ is *rational* if for any lift F of f, $\tau(F)$ is rational.

4.2. **Dynamical properties.**

Proposition 4.4. *Let f be an orientation preserving homeomorphism of $\mathbb{S}^1$. Then, $\tau(f)$ is rational if and only if f has a periodic point.*

Proof. We have already proved that if f has a periodic point, and F is any lift of f as above, then $\tau(F)$ is rational. So we must prove the converse.

Let F be a lift of f.

Note that for any integers m and k, we have $\tau(F^m + k) = m\tau(F) + k$ where $(F^m + k)(x)$ is defined to be $F^m(x) + k$ for all x.

Assume that $\tau(F) = \dfrac{p}{q}$ for some integers p and $q \neq 0$. Then, $q\tau(F) - p = 0$, so that map $G := F^q - p$ has rotation number 0.

If $G(x) - x = 0$ for some $x \in \mathbb{R}$, then G has a fixed point x. Hence f has a periodic point (of period q).

Now we suppose that G has no fixed point. Then either $G(x) - x > 0$ for all x or $G(x) - x < 0$ for all x. By translating by the lift F by an integer, we may assume that $G(x) - x > 0$. Consider $\{G^n(0)\}$ for $n > 0$. By Claim 4.5 below $\{G^n(0)\}$ is bounded above by 1. Clearly the sequence is monotone. So $\{G^n(0)\}$ must converge to some y. It follows that

$$G(y) = G(\lim_{n\to\infty} G^n(0)) = \lim_{n\to\infty} G(G^n(0)) = \lim_{n\to\infty} G^{n+1}(0) = y,$$

contradicting the supposition that G has no fixed point. $\square$

Claim 4.5. *If $G(x) - x > 0$ for all x, then the sequence $\{G^n(0)\}$ is bounded above by* 1.

Proof. Suppose there exists a number k such that $G^k(0) > 1$. Then

$$G^{2k}(0) = G^k(G^k(0)) > G^k(1) = G^k(0+1) = G^k(0)+1 > 2.$$

Similarly, $G^{nk}(0) > n$ for all $n > 0$. Hence

$$\lim_{n\to\infty} \frac{G^{nk}(0)}{nk} \geq \frac{1}{k}$$

which would contradict $\tau(G) = 0$. □

Suppose the rotation number of f is rational, say $\tau(f) = \frac{p}{q}$. Then f^q has rotation number 0, and therefore has fixed points. In this case, $P(f) = \Omega(f) = \mathrm{Fix}(f^q)$, and for any $x \in \mathbb{S}^1$, $\alpha(x) \cup \omega(x) \subset \mathrm{Fix}(f^q)$, where $\mathrm{Fix}(f)$ denote the set of fixed points of f.

Now we consider the case that the rotation number of f is irrational.

Lemma 4.6. *Suppose the rotation number of f is irrational. For any $x \in \mathbb{S}^1$ and $m, n \in \mathbb{Z}$ with $m \neq n$, let $I = [f^m(x), f^n(x)]$. Then any forward orbit intersects I, i.e., for each $z \in \mathbb{S}^1$, there is a $k > 0$ such that $f^k(z) \in I$.*

Proof. The intervals $f^{-k(m-n)}I$ and $f^{-(k-1)(m-n)}I$ have one boundary point in common. So either $\{f^{-k(m-n)}I\}$ converge monotonically to a point on $\mathbb{S}^1$ or some finite union of them covers $\mathbb{S}^1$. Since the former case implies that f^{m-n} has a fixed point, contradiciting the fact that $\tau(f)$ is irrational, the latter must occur and the lemma is proved. $\square$

Proposition 4.7. *Suppose the rotation number of f is irrational. Then*

(1) *$\omega(x)$ is independent of x; and*
(2) *$\omega(x)$ is a perfect invariant set which is either nowhere dense or the whole circle $\mathbb{S}^1$.*

Proof. (1) Let $x, y \in \mathbb{S}^1$. Let $x_0 \in \omega(x)$. By definition, there is a sequence $n_1 < n_2 < \ldots$ such that $f^{n_i}(x) \to x_0$. Take $m_0 = 0$. We define an increasing squence $\{m_i\}$ inductively as follows. Suppose m_{i-1} is taken. We apply the the above lemma with $I = [f^{n_i}(x), f^{n_{i+1}}(x)]$ and $z = f^{m_{i-1}}(y)$ to get $k_i > 0$ such that $f^{k_i}(f^{m_{i-1}}(y)) = f^{k_i}(z) \in [f^{n_i}(x), f^{n_{i+1}}(x)]$. Then we let $m_i = m_{i-1} + k_i$. Clearly $f^{m_i}(y) \to x_0$, and therefore $x_0 \in \omega(y)$. Thus, $\omega(x) \subset \omega(y)$. Interchanging x and y, gives $\omega(y) \subset \omega(x)$.

(2) Let $E = \omega(x)$ which we have seen is independent of x. Since $\omega(x)$ is f-invariant, we only need to show that E is perfect. Take any $z \in E$. Since $E = \omega(x) = \omega(z)$, we have $z \in \omega(z)$. Then there is a sequence $n_1 < n_2 < \ldots$ such that $f^{n_i}(z) \to z$. Since $f(E) = E$, $f^{n_i}(z) \in E$. Also, since f has no periodic points, $f^{n_i}(x) \neq f^{n_{i+1}}(z)$. So z is a limit point of E, and E is perfect.

Since each orbit has the same ω-limit set E, it follows that E is the unique minimal set of f. Note that the boundary of E is a closed subset of E which is also invariant. The boundary of E is either equal to E itself, or an empty set, which means that either E is nowhere dense, or $E = \mathbb{S}^1$. $\square$

Corollary 4.8. *Let $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ be a circle rotation with an irrational angle. Then every orbit is dense in $E = \mathbb{S}^1$*

Proof. Observe that if $x_0 \in \omega(x)$, then for any $a \neq 0$, $x_0 + a \in \omega(x+a) = \omega(x)$ by the fact that the map is a rotation, and by part (1) of the proposition. Hence we must have $\omega(x) = \mathbb{S}^1$, and therefore $O(x)$ is dense in $\mathbb{S}^1$. $\square$

Note that in the case $\omega(x) \neq \mathbb{S}^1$, the complement of $\omega(x)$ is a open set. Hence it consists of infinitely many pairwise disjoint subintervals $\{I_j\}$, and f maps each interval to another. For any j, $f^n(I_j) \neq f^m(I_j)$ whenever $n \neq m$, since if otherwise there will be a periodic interval I_j and the rotation number will become rational. It follow that the intervals are wandering sets, which is called *wandering intervals*. In this case, $\Omega(f) = \omega(x)$ for any $x \in \mathbb{S}^1$.

A homeomorphism is *topologically transitive* if it has a dense orbit.

It is clear that if $\omega(x) = \mathbb{S}^1$ for some $x \in \mathbb{S}^1$, then f is topologically transitive.

Theorem 4.9 (Poitcaré Classification)**.** *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation preserving homeomorphism with irrational rotation number τ.*

(1) *If f is topologically transitive, then f is topologically conjugate to the rotation R_τ.*
(2) *If f is not topologically transitive, then R_τ is a factor of f, and the factor map $h : \mathbb{S}^1 \to \mathbb{S}^1$ can be chosen to be monotone.*

These two cases corresponding to the cases stated in Proposition 4.7. In the second case, h is constant on each wandering interval.

The next result shows that $\tau(f)$ is a topological conjugacy invariant.

Proposition 4.10. *Suppose f and h are order preserving circle homeomorphisms and $g = hfh^{-1}$. Then, $\tau(f) = \tau(g)$.*

Proof. Let F be a monotone lift of f such that $F-\mathrm{id}$ is periodic of period 1, and let H be a monotone lift of h such that $H - \mathrm{id}$ is periodic of period 1. Then, one can check that $\pi H^{-1} = h^{-1}\pi$, and $H^{-1} - \mathrm{id}$ is periodic of period 1. Further $G := HFH^{-1}$ is a lift of g such that $G - \mathrm{id}$ is periodic of period 1. Now,

$$\lim_{n\to\infty} \frac{G^n(0)}{n} = \lim_{n\to\infty} \frac{HF^nH^{-1}(0)}{n}.$$

Since $H - \mathrm{id}$ has period 1, we have that there is a real number $M > 0$ such that $|H(x) - x| \leq M$ for all $x \in \mathbb{R}$. Thus, $|G^n(0) - F^nH^{-1}(0)| =$

$|HF^nH^{-1}(0) - F^nH^{-1}(0)| \leq M$ independent of n, and

$$\tau(G) = \lim_{n\to\infty} \frac{G^n(0)}{n} = \lim_{n\to\infty} \frac{F^nH^{-1}(0)}{n} = \tau(F).$$

This gives that $\tau(f) = \tau(g)$. □

4.3. **Continuity of $\tau(f)$ and Cantor phenomena.** We shall next show that the rotation number $\tau(f)$ depends continuously on f in C^0 topology.

We consider the set $\mathrm{Homeo}(\mathbb{S}^1)$ of orientation preserving homeomorphisms of the circle $\mathbb{S}^1$. Let d denote the metric on $\mathbb{S}^1$. Define the C^0 distance d_0 between two continuous maps $f : \mathbb{S}^1 \to \mathbb{S}^1$ and $g : \mathbb{S}^1 \to \mathbb{S}^1$ to be

$$d_0(f, g) = \sup_{x\in\mathbb{S}^1} d(f(x), g(x)),$$

and then define

$$d(f, g) = \max\left\{d_0(f, g), d_0(f^{-1}, g^{-1})\right\}.$$

It is easy to see that this is a metric on $\mathrm{Homeo}(\mathbb{S}^1)$. The topology induced by d is called the C^0 topology.

Proposition 4.11. *The rotation number map $f \to \tau(f)$ is a continuous map from* $\mathrm{Homeo}(\mathbb{S}^1)$ *to* $\mathbb{S}^1$*.*

Proof. Let $1 > \epsilon > 0$. We show that if $f, g \in \mathrm{Homeo}(\mathbb{S}^1)$ are close then $|\tau(f) - \tau(g)| < \epsilon$.

Let $N > 0$ be such that $\frac{1}{N} < \epsilon$. If f is close enough to g, there will be lifts F of f and G of g such that $|F^N(x) - G^N(x)| < \epsilon$ for all $x \in [0, 1]$. Hence,

$|F^N(x)-G^N(x)| = |(F^N(x)-x)-(G^N(x)-x)| < \epsilon$ for all $x \in \mathbb{R}$ since $F^N(x) - x$ and $G^N(x) - x$ are periodic of period 1.

By the claim below we have that for any $k \in \mathbb{N}$, $F^{kN}(0) < G^{kN}(0)+k-1+\epsilon$. Dividing the inequality by kN, and letting $k \to \infty$, we get $\tau(F) \leq \tau(G) + \frac{1}{N} < \tau(G)+\epsilon$. Interchange F and G to get $\tau(G) \leq \tau(F)+\epsilon$, proving the proposition. □

Claim 4.12. *for any* $k \in \mathbb{N}$, $F^{kN}(0) < G^{kN}(0) + k - 1 + \epsilon$.

Proof. Using the facts that F^N and G^N are monotonic, $F^N(0) < G^N(0)+\epsilon$, and $G^N(x)-x$ is periodic of period 1, we have

$$\begin{aligned} F^{2N}(0) =& F^N(F^N(0)) < F^N(G^N(0)+\epsilon) < G^N(G^N(0)+\epsilon)+\epsilon \\ <& G^N(G^N(0)+1)+\epsilon = G^{2N}(0)+1+\epsilon. \end{aligned}$$

This proves the claim for $k = 2$. For $k = 1$ it is clear. Assume, inductively, that it is true for k. Then,

$$\begin{aligned} & F^{(k+1)N}(0) = F^N(F^{kN}(0)) < F^N(G^{kN}(0)+k-1+\epsilon) \\ =& F^N(G^{kN}(0)+\epsilon)+k-1 < G^N(G^{kN}(0)+\epsilon)+\epsilon+k-1 \\ <& G^N(G^{kN}(0)+1)+\epsilon+k-1 < G^{(k+1)N}(0)+k+\epsilon \end{aligned}$$

which is the claim for $k+1$. So, by induction, the claim is proved. □

Define "$<$" on $\mathbb{S}^1$ by $[x] < [y]$ if $y - x \in (0, 1/2)$ (mod 1) and define a partial ordering "$\prec$" on the collection of orientation-preserving circle homeomorphisms by $f_0 \prec f_1$ if $f_0(x) < f_1(x)$ for all $x \in \mathbb{S}^1$.

Notice that neither of these orderings is transitive. Indeed, $[0] < [1/3] < [2/3] < [0]$ and correspondingly $R_0 \prec R_{1/3} \prec R_{2/3} \prec R_0$, where R_α is the rotation.

It is easy to see that if $f_1 \prec f_2$, then $\tau(f_1) \leq \tau(f_2)$.

Proposition 4.13. *Let $f: \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving homeomorphism with rational rotation number $\tau(f)$.*

(i) *If $\tau(f) \notin \mathbb{Q}$, then $f \prec \bar{f}_1$ implies $\tau(f) < \tau(\bar{f}_1)$.*

(ii) *If $\tau(f) = p/q \in \mathbb{Q}$ and f has some non-periodic points, then all sufficiently nearby perturbations $\bar{f}$ with $\bar{f} \prec f$ or $f \prec \bar{f}$ (or both) have the same rotation number p/q.*

(iii) *If $\tau(f) \in \mathbb{Q}$ and all points of a map f are periodic, then the rotation number is strictly increasing at f.*

Definition 4.2. *A monotone continuous function $\phi: [0,1] \to \mathbb{R}$ (or $\phi: [0,1] \to \mathbb{S}^1$) is called a* devil's staircase *if there exists a family $\{I_\alpha\}_{\alpha \in A}$ of disjoint closed subintervals of [0,1] of nonzero length with dense union such that ϕ takes distinct constant values on these subintervals.*

Based on Proposition 4.13 we have the following.

Proposition 4.14. *Suppose that $(f_t)_{t\in[0,1]}$ is a monotone continuous family of orientation-preserving* *circle homeomorphisms, each of which has some nonperiodic points. Then $\tau : t \to \tau(f_t)$ is a devil's staircase.*

4.4. **Circle diffeomorphisms.** A partition on the interval $[0, 1]$ is given by $0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = 1$. A partition on the unit circle $\mathbb{S}^1$ can be regarded as a partition on the interval $[0, 1]$, wirh 0 and 1 being identified.

For a function $\phi : [0, 1] \to \mathbb{R}$, the total variation is given by

$$\mathrm{Var}(\phi) = \sup \sum_k = 1^n |\phi(x_k) - \phi(x_{k-1})|,$$

where supremum is taken over all partitions.

Theorem 4.15 (Denjoy)**.** *Let f be an orientation preserving C^1 diffeomorphism of the circle with irretional rotation number $\tau = \tau(f)$. If f' has bounded variation, then f is topologically conjugate to the rotation R_τ.*

Theorem 4.16 (Denjoy Example)**.** *For any irretional rotation number $\tau \in (0, 1)$, there exists a nontransitive C^1 orientation preserving diffeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$.*