4. Circle Homeomorphisms

4.1. Rotation numbers. Let $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be an orientation preserving homeomorphism. Let $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$ be the map $\pi(t) = \exp(2\pi it)$.

**Lemma 4.1.** There is a continuous map $F : \mathbb{R} \rightarrow \mathbb{R}$ such that

(i) $\pi F = f \pi$;

(ii) $F$ is monotone increasing;

(iii) $F - \text{id}$ is periodic with period 1.

Moreover, any two such maps differ by an integer translation.

**Proof.** Define $F(0)$ to be any number in the set $\pi^{-1} f(\pi(0))$. Let $U$ and $V$ be neighborhoods of 0 and $F(0)$ respectively that have length less than 1. Note that $\pi|_V : V \rightarrow \pi(V)$ is a homeomorphism. For any $t \in U$, define $F(t) = (\pi|_V)^{-1} \circ f(\pi(t))$ whenever it is defined. Then $F$ is extended to a neighborhoods $U' \subseteq U$. Using the same way we can extend the definition of $F$ to $\mathbb{R}$. It is easy to check (i)-(iii).

Suppose $G : \mathbb{R} \rightarrow \mathbb{R}$ is also a such map. Then by (i) we have that for any $t \in \mathbb{R}$, $\pi(G(t)) = f(\pi(t)) = \pi(F(t))$. That is, there exists an integer $n = n_t$ such that $G(t) = F(t) + n_t$. Since both $F$ and $G$ are continuous, and $n_t$ must be a integer, it must be independent of $t$.

Note that (i) implies that $F$ is a homeomorphism. We call such an $F$ a **lift** of $f$. 

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Proposition 4.2. Given $F$ as above, the limit

$$\tau(F) := \lim_{n \to \infty} \frac{F^n(x)}{n}$$

exists for each $x \in \mathbb{R}$, and and is independent of $x$.

Proof. (1) Independence of $x$:

Since $F(x + 1) = F(x) + 1$ for all $x$, it follows that $F^n(x + 1) = F^n(x) + 1$ for all $x$ and $n$. Now, suppose that $x \leq y \leq x + 1 \leq y + 1$. Since $F^n$ is monotone increasing, using $F^n(x + 1) = F^n(x) + 1$, we have

$$\frac{F^n(x)}{n} \leq \frac{F^n(y)}{n} \leq \frac{F^n(x + 1)}{n} \leq \frac{F^n(y + 1)}{n}.$$ 

This implies that if the limit $\lim_{n \to \infty} \frac{F^n(x)}{n}$ exists, then so does $\lim_{n \to \infty} \frac{F^n(y)}{n}$, and they are equal.

(2) Existence if $f$ has a periodic point:

Let $x$ be a periodic point of period $m$, and let $y \in \mathbb{R}$ be such that $\pi(y) = x$. Then there is an integer $p$ such that $F^m(y) = y + p$. Then, $F^{nm}(y) = y + np$. So

$$\lim_{n \to \infty} \frac{F^{nm}(x)}{nm} = \lim_{n \to \infty} \frac{y + np}{nm} = \frac{p}{m}$$

Now, for any integer $k$, let $k = rm + q$ with $0 \leq q < m$. Then,

$$\frac{F^k(y)}{k} = \frac{F^k(y) - F^{rm}(y) + F^{rm}(y)}{k}$$
and
\[
\left| \frac{F^k(y) - F^{rm}(y)}{k} \right| \leq \frac{M}{k}
\]
where \( M = \max_{0 \leq q < m} |F^q(y) - y| \). Thus,
\[
\lim_{k \to \infty} \frac{F^k(x)}{k} = \lim_{k \to \infty} \frac{F^{rm}(x)}{k} = \lim_{r \to \infty} \frac{F^{rm}(y)}{rm} = \frac{p}{m}.
\]
Thus, the limit exists if \( f \) has a periodic point.

(3) Existence if \( f \) has no periodic points:
This implies that \( F^m(x) - x \) is not an integer for any \( m > 0 \) and any \( x \in \mathbb{R} \). Let \( p_m \) be an integer such that
\[
(4.1) \quad p_m < F^m(0) < p_m + 1
\]
Therefore, for all \( x \in \mathbb{R} \), \( p_m < F^m(x) - x < p_m + 1 \), since if otherwise, then by the Intermediate Value Theorem, we have \( F^m(y) - y = p_m \) or \( F^m(y) - y = p_m + 1 \) for some \( y \), which is a contradiction. Hence, for \( 1 \leq i \leq n \), \( p_m < F^{im}(0) - F^{(i-1)m}(0) < p_m + 1 \). Adding together these inequalities for \( i = 1, \ldots, n \), the middle terms telescope, and we get
\[
(4.2) \quad np_m < F^{nm}(0) < n(p_m + 1)
\]
Dividing (4.1) by \( m \) and (4.1) by \( mn \), we get that \( \frac{F^{nm}(0)}{mn} \) and \( \frac{F^{m}(0)}{m} \) are both in the interval \( \left( \frac{p_m}{m}, \frac{p_m + 1}{m} \right) \). So
\[
\left| \frac{F^{nm}(0)}{mn} - \frac{F^{m}(0)}{m} \right| \leq \frac{1}{m}.
\]
Interchanging the roles of $m$ and $n$, we get

$$\left| \frac{F^{nm}(0)}{mn} - \frac{F^n(0)}{n} \right| \leq \frac{1}{n},$$

and, hence,

$$\left| \frac{F^m(0)}{m} - \frac{F^n(0)}{n} \right| \leq \frac{1}{m} + \frac{1}{n}.$$ 

Hence, the sequence $\left\{ \frac{F^n(0)}{n} \right\}$ is a Cauchy sequence, and thus has a limit. \qed

**Lemma 4.3.** Let $F$ and $G$ are both lift of $f$, then there exists $p \in \mathbb{Z}$ such that $\tau(G) = \tau(F) + p$.

**Proof.** Since $F$ and $G$ are both lift of $f$, then there exists $p \in \mathbb{Z}$ such that $G(x) = F(x) + p$ for any $x \in \mathbb{R}$. So we have $G^2(x) = G(G(x)) = F(F(x) + p) + p = F(x) + 2p$, and for each $n > 0$, $G^n(x) = F^n(x) + np$. Hence,

$$\tau(G) = \lim_{n \to \infty} \frac{G^n(x)}{n} = \lim_{n \to \infty} \frac{F^n(x) + np}{n} = \tau(F) + p. \quad \Box$$

The above lemma says that $\tau(f)$ is independent of the choice of the lift $F$.

**Definition 4.1.** The number $\tau(f) := \pi \tau(F)$ is called the rotation number of $f$.

We say that $\tau(f)$ is rational if for any lift $F$ of $f$, $\tau(F)$ is rational.
4.2. Dynamical properties.

**Proposition 4.4.** Let $f$ be an orientation preserving homeomorphism of $S^1$. Then, $\tau(f)$ is rational if and only if $f$ has a periodic point.

*Proof.* We have already proved that if $f$ has a periodic point, and $F$ is any lift of $f$ as above, then $\tau(F)$ is rational. So we must prove the converse.

Let $F$ be a lift of $f$.

Note that for any integers $m$ and $k$, we have $\tau(F^m + k) = m\tau(F) + k$ where $(F^m + k)(x)$ is defined to be $F^m(x) + k$ for all $x$.

Assume that $\tau(F) = \frac{p}{q}$ for some integers $p$ and $q \neq 0$. Then, $q\tau(F) - p = 0$, so that map $G := F^q - p$ has rotation number 0.

If $G(x) - x = 0$ for some $x \in \mathbb{R}$, then $G$ has a fixed point $x$. Hence $f$ has a periodic point (of period $q$).

Now we suppose that $G$ has no fixed point. Then either $G(x) - x > 0$ for all $x$ or $G(x) - x < 0$ for all $x$. By translating by the lift $F$ by an integer, we may assume that $G(x) - x > 0$. Consider $\{G^n(0)\}$ for $n > 0$. By Claim 4.5 below $\{G^n(0)\}$ is bounded above by 1. Clearly the sequence is monotone. So $\{G^n(0)\}$ must converge to some $y$. It follows that $G(y) = G(\lim_{n \to \infty} G^n(0)) = \lim_{n \to \infty} G(G^n(0)) = \lim_{n \to \infty} G^{n+1}(0) = y$, contradicting the supposition that $G$ has no fixed point. \qed
Claim 4.5. If $G(x) - x > 0$ for all $x$, then the sequence $\{G^n(0)\}$ is bounded above by 1.

Proof. Suppose there exists a number $k$ such that $G^k(0) > 1$. Then

$$G^{2k}(0) = G^k(G^k(0)) > G^k(1) = G^k(0+1) = G^k(0)+1 > 2.$$  

Similarly, $G^{nk}(0) > n$ for all $n > 0$. Hence

$$\lim_{n \to \infty} \frac{G^{nk}(0)}{nk} \geq \frac{1}{k}$$

which would contradict $\tau(G) = 0$. $\square$

Suppose the rotation number of $f$ is rational, say $\tau(f) = \frac{p}{q}$. Then $f^q$ has rotation number 0, and therefore has fixed points. In this case, $P(f) = \Omega(f) = \text{Fix}(f^q)$, and for any $x \in S^1$, $\alpha(x) \cup \omega(x) \subset \text{Fix}(f^q)$, where $\text{Fix}(f)$ denote the set of fixed points of $f$.

Now we consider the case that the rotation number of $f$ is irrational.

Lemma 4.6. Suppose the rotation number of $f$ is irrational. For any $x \in S^1$ and $m,n \in \mathbb{Z}$ with $m \neq n$, let $I = [f^m(x), f^n(x)]$. Then any forward orbit intersects $I$, i.e., for each $z \in S^1$, there is a $k > 0$ such that $f^k(z) \in I$.

Proof. The intervals $f^{-k(m-n)}I$ and $f^{-(k-1)(m-n)}I$ have one boundary point in common. So either $\{f^{-k(m-n)}I\}$ converge monotonically to a point on $S^1$ or some finite union of them covers $S^1$. Since the former case
implies that \( f^{m-n} \) has a fixed point, contradicting the fact that \( \tau(f) \) is irrational, the latter must occur and the lemma is proved. \( \square \)

**Proposition 4.7.** Suppose the rotation number of \( f \) is irrational. Then

(1) \( \omega(x) \) is independent of \( x \); and

(2) \( \omega(x) \) is a perfect invariant set which is either nowhere dense or the whole circle \( S^1 \).

**Proof.** (1) Let \( x, y \in S^1 \). Let \( x_0 \in \omega(x) \). By definition, there is a sequence \( n_1 < n_2 < \ldots \) such that \( f^{n_i}(x) \to x_0 \). Take \( m_0 = 0 \). We define an increasing sequence \( \{m_i\} \) inductively as follows. Suppose \( m_{i-1} \) is taken. We apply the above lemma with \( I = [f^{n_i}(x), f^{n_{i+1}}(x)] \) and \( z = f^{m_{i-1}}(y) \) to get \( k_i > 0 \) such that \( f^{k_i}(f^{m_{i-1}}(y)) = f^{k_i}(z) \in [f^{n_i}(x), f^{n_{i+1}}(x)] \). Then we let \( m_i = m_{i-1} + k_i \). Clearly \( f^{m_i}(y) \to x_0 \), and therefore \( x_0 \in \omega(y) \). Thus, \( \omega(x) \subset \omega(y) \). Interchanging \( x \) and \( y \), gives \( \omega(y) \subset \omega(x) \).

(2) Let \( E = \omega(x) \) which we have seen is independent of \( x \). Since \( \omega(x) \) is \( f \)-invariant, we only need to show that \( E \) is perfect. Take any \( z \in E \). Since \( E = \omega(x) = \omega(z) \), we have \( z \in \omega(z) \). Then there is a sequence \( n_1 < n_2 < \ldots \) such that \( f^{n_i}(z) \to z \). Since \( f(E) = E, f^{n_i}(z) \in E \). Also, since \( f \) has no periodic points, \( f^{n_i}(x) \neq f^{n_{i+1}}(z) \). So \( z \) is a limit point of \( E \), and \( E \) is perfect.
Since each orbit has the same \( \omega \)-limit set \( E \), it follows that \( E \) is the unique minimal set of \( f \). Note that the boundary of \( E \) is a closed subset of \( E \) which is also invariant. The boundary of \( E \) is either equal to \( E \) itself, or an empty set, which means that either \( E \) is nowhere dense, or \( E = \mathbb{S}^1 \). \(
abla\)

**Corollary 4.8.** Let \( R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1 \) be a circle rotation with an irrational angle. Then every orbit is dense in \( E = \mathbb{S}^1 \)

*Proof.* Observe that if \( x_0 \in \omega(x) \), then for any \( a \neq 0 \), \( x_0 + a \in \omega(x + a) = \omega(x) \) by the fact that the map is a rotation, and by part (1) of the proposition. Hence we must have \( \omega(x) = \mathbb{S}^1 \), and therefore \( O(x) \) is dense in \( \mathbb{S}^1 \). \(
abla\)

Note that in the case \( \omega(x) \neq \mathbb{S}^1 \), the complement of \( \omega(x) \) is a open set. Hence it consists of infinitely many pairwise disjoint subintervals \( \{I_j\} \), and \( f \) maps each interval to another. For any \( j \), \( f^n(I_j) \neq f^m(I_j) \) whenever \( n \neq m \), since if otherwise there will be a periodic interval \( I_j \) and the rotation number will become rational. It follow that the intervals are wandering sets, which is called *wandering intervals*. In this case, \( \Omega(f) = \omega(x) \) for any \( x \in \mathbb{S}^1 \).

A homeomorphism is *topologically transitive* if it has a dense orbit.

It is clear that if \( \omega(x) = \mathbb{S}^1 \) for some \( x \in \mathbb{S}^1 \), then \( f \) is topologically transitive.
Theorem 4.9 (Poincaré Classification). Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism with irrational rotation number $\tau$.

1. If $f$ is topologically transitive, then $f$ is topologically conjugate to the rotation $R_{\tau}$.
2. If $f$ is not topologically transitive, then $R_{\tau}$ is a factor of $f$, and the factor map $h : S^1 \to S^1$ can be chosen to be monotone.

These two cases corresponding to the cases stated in Proposition 4.7. In the second case, $h$ is constant on each wandering interval.

The next result shows that $\tau(f)$ is a topological conjugacy invariant.

Proposition 4.10. Suppose $f$ and $h$ are order preserving circle homeomorphisms and $g = hfh^{-1}$. Then, $\tau(f) = \tau(g)$.

Proof. Let $F$ be a monotone lift of $f$ such that $F - \text{id}$ is periodic of period 1, and let $H$ be a monotone lift of $h$ such that $H - \text{id}$ is periodic of period 1. Then, one can check that $\pi H^{-1} = h^{-1} \pi$, and $H^{-1} - \text{id}$ is periodic of period 1. Further $G := HFH^{-1}$ is a lift of $g$ such that $G - \text{id}$ is periodic of period 1. Now,

$$\lim_{n \to \infty} \frac{G^n(0)}{n} = \lim_{n \to \infty} \frac{HF^n H^{-1}(0)}{n}.$$ 

Since $H - \text{id}$ has period 1, we have that there is a real number $M > 0$ such that $|H(x) - x| \leq M$ for all $x \in \mathbb{R}$. Thus, $|G^n(0) - F^n H^{-1}(0)| =$
\[ |HF^n H^{-1}(0) - F^n H^{-1}(0)| \leq M \] independent of \( n \), and
\[
\tau(G) = \lim_{n \to \infty} \frac{G^n(0)}{n} = \lim_{n \to \infty} \frac{F^n H^{-1}(0)}{n} = \tau(F).
\]
This gives that \( \tau(f) = \tau(g) \).

4.3. **Continuity of \( \tau(f) \) and Cantor phenomena.** We shall next show that the rotation number \( \tau(f) \) depends continuously on \( f \) in \( C^0 \) topology.

We consider the set \( \text{Homeo}(\mathbb{S}^1) \) of orientation preserving homeomorphisms of the circle \( \mathbb{S}^1 \). Let \( d \) denote the metric on \( \mathbb{S}^1 \). Define the \( C^0 \) distance \( d_0 \) between two continuous maps \( f : \mathbb{S}^1 \to \mathbb{S}^1 \) and \( g : \mathbb{S}^1 \to \mathbb{S}^1 \) to be
\[
d_0(f, g) = \sup_{x \in \mathbb{S}^1} d(f(x), g(x)),
\]
and then define
\[
d(f, g) = \max \{ d_0(f, g), d_0(f^{-1}, g^{-1}) \}.
\]
it is easy to see that this is a metric on \( \text{Homeo}(\mathbb{S}^1) \).

The topology induced by \( d \) is called the \( C^0 \) topology.

**Proposition 4.11.** The rotation number map \( f \to \tau(f) \) is a continuous map from \( \text{Homeo}(\mathbb{S}^1) \) to \( \mathbb{S}^1 \).

**Proof.** Let \( 1 > \epsilon > 0 \). We show that if \( f, g \in \text{Homeo}(\mathbb{S}^1) \) are close then \( |\tau(f) - \tau(g)| < \epsilon \).

Let \( N > 0 \) be such that \( \frac{1}{N} < \epsilon \). If \( f \) is close enough to \( g \), there will be lifts \( F \) of \( f \) and \( G \) of \( g \) such that \( |F^N(x) - G^N(x)| < \epsilon \) for all \( x \in [0, 1] \). Hence,
\[ |F^N(x) - G^N(x)| = |(F^N(x) - x) - (G^N(x) - x)| < \epsilon \]

for all \( x \in \mathbb{R} \) since \( F^N(x) - x \) and \( G^N(x) - x \) are periodic of period 1.

By the claim below we have that for any \( k \in \mathbb{N} \),

\[ F^{kN}(0) < G^{kN}(0) + k - 1 + \epsilon. \]

Dividing the inequality by \( kN \), and letting \( k \to \infty \), we get

\[ \frac{1}{N} < \tau(G) + \epsilon. \]

Interchange \( F \) and \( G \) to get \( \tau(G) \leq \tau(F) + \epsilon \), proving the proposition.

\[ \square \]

**Claim 4.12.** for any \( k \in \mathbb{N} \), \( F^{kN}(0) < G^{kN}(0) + k - 1 + \epsilon \).

**Proof.** Using the facts that \( F^N \) and \( G^N \) are monotonic, \( F^N(0) < G^N(0) + \epsilon \), and \( G^N(x) - x \) is periodic of period 1, we have

\[ F^{2N}(0) = F^N(F^N(0)) < F^N(G^N(0) + \epsilon) < G^N(G^N(0) + \epsilon) + \epsilon < G^N(G^N(0) + 1) + \epsilon = G^{2N}(0) + 1 + \epsilon. \]

This proves the claim for \( k = 2 \). For \( k = 1 \) it is clear.

Assume, inductively, that it is true for \( k \). Then,

\[ F^{(k+1)N}(0) = F^N(F^{kN}(0)) < F^N(G^{kN}(0) + k - 1 + \epsilon) \]

\[ = F^N(G^{kN}(0) + \epsilon) + k - 1 < G^N(G^{kN}(0) + \epsilon) + \epsilon + k - 1 \]

\[ < G^N(G^{kN}(0) + 1) + \epsilon + k - 1 < G^{(k+1)N}(0) + k + \epsilon \]

which is the claim for \( k + 1 \). So, by induction, the claim is proved.

\[ \square \]

Define “<” on \( S^1 \) by \([x] < [y] \) if \( y - x \in (0, 1/2) \) (mod 1) and define a partial ordering “≺” on the
collection of orientation-preserving circle homeomorphisms by $f_0 \prec f_1$ if $f_0(x) < f_1(x)$ for all $x \in \mathbb{S}^1$.

Notice that neither of these orderings is transitive. Indeed, $[0] < [1/3] < [2/3] < [0]$ and correspondingly $R_0 \prec R_{1/3} \prec R_{2/3} \prec R_0$, where $R_\alpha$ is the rotation.

It is easy to see that if $f_1 \prec f_2$, then $\tau(f_1) \leq \tau(f_2)$.

**Proposition 4.13.** Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving homeomorphism with rational rotation number $\tau(f)$.

(i) If $\tau(f) \notin \mathbb{Q}$, then $f \prec \bar{f}_1$ implies $\tau(f) < \tau(f_1)$.

(ii) If $\tau(f) = p/q \in \mathbb{Q}$ and $f$ has some non-periodic points, then all sufficiently nearby perturbations $\bar{f}$ with $\bar{f} \prec f$ or $f \prec \bar{f}$ (or both) have the same rotation number $p/q$.

(iii) If $\tau(f) \in \mathbb{Q}$ and all points of a map $f$ are periodic, then the rotation number is strictly increasing at $f$.

**Definition 4.2.** A monotone continuous function $\phi : [0, 1] \to \mathbb{R}$ (or $\phi : [0, 1] \to \mathbb{S}^1$) is called a devil’s staircase if there exists a family $\{I_\alpha\}_{\alpha \in A}$ of disjoint closed subintervals of $[0,1]$ of nonzero length with dense union such that $\phi$ takes distinct constant values on these subintervals.

Based on Proposition 4.13 we have the following.

**Proposition 4.14.** Suppose that $(f_t)_{t \in [0,1]}$ is a monotone continuous family of orientation-preserving
circle homeomorphisms, each of which has some nonperiodic points. Then \( \tau : t \to \tau(f_t) \) is a devil’s staircase.

4.4. **Circle diffeomorphisms.** A partition on the interval \([0, 1]\) is given by \(0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = 1\). A partition on the unit circle \(S^1\) can be regarded as a partition on the interval \([0, 1]\), with 0 and 1 being identified.

For a function \(\phi : [0, 1] \to \mathbb{R}\), the total variation is given by

\[
\text{Var}(\phi) = \sup \sum_{k=1}^{n} 1^n |\phi(x_k) - \phi(x_{k-1})|,
\]

where supremum is taken over all partitions.

**Theorem 4.15** (Denjoy). Let \(f\) be an orientation preserving \(C^1\) diffeomorphism of the circle with irrational rotation number \(\tau = \tau(f)\). If \(f'\) has bounded variation, then \(f\) is topologically conjugate to the rotation \(R_\tau\).

**Theorem 4.16** (Denjoy Example). For any irrational rotation number \(\tau \in (0, 1)\), there exists a nontransitive \(C^1\) orientation preserving diffeomorphism \(f : S^1 \to S^1\).