2. Contracting maps and fixed point theorems

Let X be a metric space and let $T : X \to X$ be a mapping. Recall that a fixed point p of T is a point $p \in X$ such that $T(p) = p$.

A self-map T of a metric space X is called a *contraction* (or contraction map or mapping) if there is a constant $0 < \lambda < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$. Thus, $T : X \to X$ is a contraction if and only it is Lipschitz with Lipschitz constant less than 1.

Theorem 2.1 (Contraction Mapping Theorem)**.** *Suppose X is a complete metric space and $T : X \to X$ is a contraction map. Then T has a unique fixed point $\bar{x}$ in X.*

Moreover, if x is any point in $\mathcal{F}$, then the sequence of iterates x, Tx, $T^2x, \ldots$ converges to $\bar{x}$ exponentially fast.

Proof. (Uniqueness) If $0 < \lambda < 1$ is the contraction constant for T and $Tx = x, Ty = y$, then

$$d(x, y) = d(Tx, Ty) \leq \lambda d(x, y)$$

which implies that $d(x, y) = 0$. This in turn implies that $x = y$.

(Existence) Take any $x \in X$ and let $x_0 = x$, $x_i = T^i x$ for $i > 0$. Then,

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}) \leq \ldots \leq \lambda^n d(x_1, x_0) \quad \forall n \geq 1.$$

Thus, for $m > n$,

$$\begin{aligned} d(x_m, x_n) \leq& d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n) \\ \leq& (\lambda^{m-1} + \lambda^{m-2} + \ldots + \lambda^n) d(x_1, x_0) \\ =& \frac{\lambda^n(1 - \lambda^{m-n})}{1 - \lambda} d(x_1, x_0) \leq C\lambda^n d(x_1, x_0), \end{aligned} \tag{2.1}$$

where $C = 1/(1 - \lambda)$.

This implies that the sequence $\{x_i\}_{i=1,2,\ldots}$ is a Cauchy sequence. By completeness of X, it converges, say to an element $\bar{x}$ of X. But, since T is continuous,

$$T(\bar{x}) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = \bar{x},$$

so, $T(\bar{x}) = \bar{x}$. This proves the existence.

(Convergence rates) Since (2.1) holds for any $m \geq n$, let $m \to \infty$ we get

$$d(\bar{x}, x_n) \leq C\lambda^n d(x_1, \bar{x}).$$

The fact $\lambda \in (0, 1)$ gives that the convergence is exponential. $\square$

So in a dynamical system (X, T), if X is a complete metric space and T is contracting map, then there is a unique fixed point $\bar{x}$, that is, $T(\bar{x}) = \bar{x}$. For any other point $x \in X$, we have that

$$\lim_{n\to\infty} T^n x = \bar{x}$$

exponentially fast.

The preceding theorem gives a useful sufficient condition for the existence of fixed points in a wide variety of situations. It is frequently useful to know when such fixed points depend continuously on parameters. This leads us to the next result.

Definition 2.2. *Let Λ be a topological space (e.g. a metric space), and let X be a complete metric space. A map T from Λ into the space of maps $\mathcal{M}(X, X)$ is called a* continuous family of self-maps of X *if the map $\widehat{T}(\lambda, x) = T(\lambda)(x)$ is continuous as a map from the product space $\Lambda \times X$ to X.*

The map T is called a uniform family of contractions *on X if it is a continuous family of self-maps of X and there is a constant $0 < \alpha < 1$ such that*

$$d(\widehat{T}(\lambda, x), \widehat{T}(\lambda, y)) \leq \alpha d(x, y)$$

for all $x, y \in X, \lambda \in \Lambda$.

Thus, the continuous family is a uniform family of contractions if and only if all the maps in the family have the same upper bound $\alpha < 1$ for their Lipschitz constants.

Given the family $\widehat{T}$ as above, we define the map $T_\lambda : X \to X$ by

$$T_\lambda(x) = T(\lambda)(x) = \widehat{T}(\lambda, x)$$

Theorem 2.3. *If $T : \Lambda \to \mathcal{M}(X, X)$ is a uniform family of contractions on X, then each map T_λ has a unique fixed point x_λ which depends continuously on λ. That is, the map $\lambda \to \bar{x}_\lambda$ is a continuous map from Λ into X.*

Proof. Let $g(\lambda)$ be the fixed point of the map T_λ which exists since the map T_λ is a contraction.

For $\lambda_1, \lambda_2 \in \Lambda$, we have

$$\begin{aligned} d(g(\lambda_1), g(\lambda_2)) =& d(T_{\lambda_1} g(\lambda_1), T_{\lambda_2} g(\lambda_2)) \\ \leq& d(T_{\lambda_1} g(\lambda_1), T_{\lambda_1} g(\lambda_2)) + d(T_{\lambda_1} g(\lambda_2), T_{\lambda_2} g(\lambda_2)) \\ \leq& \alpha d(g(\lambda_1), g(\lambda_2)) + d(T_{\lambda_1} g(\lambda_2), T_{\lambda_2} g(\lambda_2)). \end{aligned}$$

This implies that

$$d(g(\lambda_1), g(\lambda_2)) \leq (1 - \alpha)^{-1} d(T_{\lambda_1} g(\lambda_2), T_{\lambda_2} g(\lambda_2)).$$

Since the map $\lambda \to T_\lambda g(\lambda_2)$ is continuous for fixed λ_2, we see that $\lambda \to g(\lambda)$ is continuous. $\square$

Recall that a *normed linear (vector) space* is an ordered pair $(X, \| \cdot \|)$ where X is a vector space and $\| \cdot \| : X \to \mathbb{R}$ is a real-valued function on X such that

(i) $\| x \| \geq 0 \; \forall x$ and $\| x \| = 0$ iff $x = 0$ for $x \in X$;
(ii) $\| \alpha x \| = \| \alpha \| \| x \|$ for $\alpha \in \mathbb{R}, x \in X$;
(iii) $\| x + y \| \leq \| x \| + \| y \| \; \forall \; x, y \in X$.

A normed linear space $(X, \| \cdot \|)$ is called a *Banach space* if it is a complete metric space with respect to the metric $d(x, y) = \| x - y \|$ induced by the norm.

Let X be a metric space and Y be a Banach space. For a bounded function g from X to Y, the *sup norm*, or C^0 *norm*, of g is given by

$$\|g\| = \|g\|_0 = \sup_{x \in X} \| g(x) \|.$$

Let X and and Y be Banach spaces. For a linear map, or a linear operator, f from X to Y, the norm of f is given by

$$\|f\| = \sup_{|x| \leq 1} \| f(x) \|.$$

It is known that $f : X \to Y$ is *bounded* if $\|f\| < \infty$.

Let M be a manifold, and $f : M \to M$ be differential. Then for any $x \in M$, the differential Df_x is a linear map from $T_x M$ to $T_{f(x)} M$. The C^1 *norm* of f is given by

$$\|f\|_1 = \sup_{x \in M} \|Df_x\|.$$

However, if M is a normed space, the C^1 *norm* of f is also given by

$$\|f\|_1 = \sup\{\|f\|, \|Df_x\| : x \in M\}.$$

Example 2.4. *Let $X = \mathbb{R}^n$, and $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with $\|A\| < 1$. Then the dynamical system $(\mathbb{R}^n, A)$ is contracting with a unique fixed point 0.*

Furthere, let $g : \mathbb{R}^n \to \mathbb{R}^n$ be an map with $\|g\|_0, \|g\|_1 < \infty$. Then for $f_\epsilon(x) = Ax + \epsilon g(x)$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in [0, \epsilon_0]$, the system $(\mathbb{R}^n, f_\epsilon)$ is contracting with a unique fixed point p_ϵ close to 0 by the above theorem.

Lemma 2.5. *Let f be a continuously differentiable map on $X = \mathbb{R}^n$ such that $\|Df_x\| \leq r < 1$ for all $x \in X$. Then for any $x, y \in X$, $|f(x) - f(y)| \leq r|x - y|$.*

Proof. Let $u(t) = tx + (1-t)y$. Then $f(u(0)) = x$ and $f(u(1)) = y$. Also,

$$\frac{d}{dt} f(u(t)) = \frac{\partial f}{\partial u}\frac{du}{dt} = Df_u(x-y).$$

By the Fundamental Theorem of Calculus, we have

$$\begin{aligned} |f(x) - f(y)| &= \Big| \int_0^1 \frac{d}{dt} f(u(t)) dt \Big| = \Big| \int_0^1 Df_u dt (x-y) \Big| \\ &\le \Big| \int_0^1 Df_u dt \Big| |x-y| \le r|x-y| \end{aligned}$$

□

Lemma 2.6. *Let f be a continuously differentiable map on $X = \mathbb{R}^n$ with a fixed point $\bar{x}$ such that all eigenvalues of $Df_{\bar{x}}$ have absolute value less than 1. Then there is a closed neighborhood U of $\bar{x}$ such that $f(U) \subset U$ and f is a contraction on U with respect to an adapted norm.*

Proof. It can be proved that the assumption on the eigenvalues implies that one can choose a norm that we denote by $\|\cdot\|'$ for which $\|Df\|' < 1$. Hence by continuity a small closed "ball" around $\bar{x}$ with respect to the norm $\|\cdot\|'$ can be chosen as the set U. (This ball is in fact an ellipsoid in $\mathbb{R}^n$.) □

Example 2.7. *Let $X = \mathbb{C}$, and let $f(z) = z^2$ for $z \in \mathbb{C}$. Since $f'(z) = 2z$, for any $r < 1$, f is a contraction on $B(r) := \{z \in \mathbb{Z} : \ |z| \le r\}$.*

For any analytic function $g : \mathbb{C} \to CC$, there existsxists $\epsilon_0 > 0$ such that for any $\epsilon \in [0, \epsilon_0]$, $f(z) = f_\epsilon(z) = z^2 + \epsilon g(z)$ is contracting on $B(r)$.

Example 2.8 (The Newton Method)**.** *Consider a function f on the real line and suppose that we have a reasonable guess x_0 for a root. Unless the graph intersects the x-axis at x_0, i.e., $f(x_0) = 0$, we need to improve our guess. To that end we take the tangent line and see at which point x_1 it intersects the x-axis by setting $f(x_0) + f'(x_0)(x_1 - x_0) = 0$. Thus the improved guess is*

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

It is clear that $\bar{x}$ is a root of the equation if and only if it is a fixed point of the map $F(x) := x - (f(x)/f'(x))$.

A fixed point $\bar{x}$ of a differentiable map F is said to be *superattracting* if $F'(\bar{x}) = 0$.

Proposition 2.9. *If $|f'(x)| > \delta$ and $|f''(x)| < M$ on a neighborhood of the root x^*, then $\bar{x}$ is a superattracting fixed point of $F(x)$.*

Proof. This is because $F'(x) = f(x)f''(x)/(f'(x))^2$ and $f(\bar{x}) = 0$. $\square$

Denote $\epsilon_n = |x_n - \bar{x}|$. If x_n is sufficient close to $\bar{x}$, then it can be proved that

$$\epsilon_{n+1} \leq M\epsilon_n^2, \qquad \text{where } M = \sup_{x \in I} \left| \frac{f''(x)}{f'(x)} \right|,$$

and I is a small interval containing $(\bar{x} - \epsilon_n, \bar{x} + \epsilon_n)$.

Other Fixed Point Theorems.

Theorem 2.10 (Brouwer Fixed Point Theorem)**.** *Every continuous map T of the closed unit ball in $\mathbb{R}^n$ to itself has a fixed point.*

For $n = 1$, the result can be obtained from the intermediate velue theorem.

Theorem 2.11. *Every continuous map T of a the compact interval I to itself has a fixed point.*

Proof. Suppose $I = [a, b]$, where $-\infty < a < b < \infty$. Since $f(I) \subset I$, we have $f(a) - a \geq 0$ and $f(b) - b \leq 0$. Then we use the intermediate velue theorem for $f - \mathrm{id}$. $\square$

Theorem 2.12 (Schauder Fixed Point Theorem)**.** *Every continuous self-map of a compact convex subset of a Banach space has a fixed point.*