1. Introduction

Physically, a dynamical system is an object or collection of objects in the real world which evolves in time.

Let us give some examples.

(1) A fluid in a container subjected to stirring or external influences such as changes in temperature or pressure
(2) The population at time t of a certain species of animal or plant
(3) The current through a wire (motion of electrons)
(4) The motion of a object suspended by a spring or rigid rod pendulum
(5) Molecules of a gas in a container

Mathematically, we introduce the following type of dynamical systems.

1.1. Discrete Dynamical System.

Let X be a set, and $f : X \to X$ be a self-map. A *dynmical system* is the pair (X, f).

(a) *Topological dynmical system* X is a topological space, and f is a continuous map or a homeomorphism. Usually X is taken to be a complete separable metric space.
(b) *Smooth dynmical system* X is a region of Euclidean space or a manifold topological space, and f is a differentiable map or a diffeomorphism.
(c) *Complex dynmical system* X is a complex plane $\mathbb{C}$ or higher dimensional complex space $\mathbb{C}^n$, and f is analytic or meromorphic function.
(d) *Ergodic theory* X is a measure space or probablility space, and f is measure preserving transformation.

We write $f^0 = \mathrm{id}$, the identity map, and $f^2 = f \circ f$ where $\circ$ denotes composition, and inductively for any $n > 0$, $f^n = f^{n-1} \circ f$. Here f^n is called the *nth iterate* of f. If f is invertible, then so are f^n for any $n > 0$. We denote by f^{-n} the inverse of f^n, that is, $f^{-n} = (f^n)^{-1}$. It is easy to check that for any $m, n \in \mathbb{Z}$, $f^{m+n} = f^m \circ f^n$.

For any $x_0 \in X$, we write $f^n(x_0) = x_n$. If x_0 is the state of our system at time 0, then x_n gives the state at time n.

For any $x \in X$, the set $O(x) = \{f^n(x) : n \in \mathbb{Z}\}$ is called the *orbit* of x. If f is noninvetible, then we use $O_+(x) = \{f^n(x) : n \in \mathbb{Z}_+\}$, where $\mathbb{Z}_+ := \mathbb{Z}_{\geq 0} = \{n \in \mathbb{Z} : n \geq 0\}$.

There are three type of orbits:

— *periodic*

An orbit $O_+(x)$ is periodic if there is an integer $n > 0$ such that $f^n(x) = x$. In this case, we also call the point x a *periodic point.* If $n = 1$, such a point x is called a fixed point.

— *eventually periodic*
An orbit $O_+(x)$ is called *eventually periodic* if there is a positive integer m such that $O_+(f^m(x))$ is periodic.

— *countable sequence of points.*

If f is invertible, then $O(x)$ or $O_+(x)$ cannot be eventually periodic except it is periodic.

Example. (1) *Let x_0 denote an initial amount of money (principle) deposited in a bank in which interest is paid at a rate of 5% per year. Let x_n denote the amount of money after n years.*
We have

$$x_1 = x_0 + .05x_0,$$
$$\ldots\ldots$$
$$x_{n+1} = x_n + .05x_n,$$

, for $n \geq 0$. We may use $f(x) = (1.05)x$, so that $f^n(x) = (1.05)^n x$ for each x.

(2) *Let $\mathbb{S}^1 = \{z \in \mathbb{C} :\mid z \mid= 1\}$ be the unit circle in the complex plane, and let $R_\alpha(z) = e^{i\alpha}z$ where $\alpha \in [0, 2\pi]$. We call R_α the* rotation by angle α. *We will see later that*
  (a) *if $\frac{\alpha}{2\pi}$ is rational then all orbits are periodic, and*
  (b) *if $\frac{\alpha}{2\pi}$ is irrational, then all ortits are dense. (A subset $A \subset X$ is dense if its closure is all of X).*

(3) *Let $I = [0, 1)$ and $f(x) = 2x \pmod 1$. f is noninvertible.*
  (a) *if $x \in \mathbb{Q}$, then $O_+(x)$ is periodic or eventually periodic.*
  (b) *if $x \notin \mathbb{Q}$, then $O_+(x)$ is a countable set.*

1.2. **Continuous dynamical system.** Let X be a set. A *semi-flow* on X is a map $\phi : \mathbb{R}_+ \times X \to X$ such that

(1) $\phi(0, x) = x$ for all $x \in X$, and
(2) $\phi(s + t, x) = \phi(s, \phi(t, x))$ for all $x \in X$, and $s, t \in \mathbb{R}_+$.

If the map ϕ is defined for all $t \in \mathbb{R}$ and satifies the preceding two properties, then it is called a *flow* in X.

The *orbit* of a point xfor semiflow and flow are the sets $O_+(x) = \{\phi(t, x) : t \in \mathbb{R}_+\}$ and $O(x) = \{\phi(t, x) : t \in \mathbb{R}\}$ respectively. We will mainly consider semi-flows which are actually flows. So we only describe the orbits of flows. There are three kinds.

— *fixed (or critical) orbit:* An orbit $O(x)$ is *fixed* or *critical* if $\phi(t,x) = x$ for all t. As above we also call the point x a *fixed* or *critical* point of ϕ.

— *periodic:* An orbit $O(x)$ is called *periodic* if, there is a real $\tau > 0$ such that $\phi(t+\tau, x) = \phi(t,x)$ for all $t \in \mathbb{R}$, and $\phi(s,x) \neq x$ for any $0 < s < \tau$. In this case, we call τ the *period* of x.

— all other orbits, these are in 1-1 correspondence with the whole set of real numbers $\mathbb{R}$.

Example. (4) *Let X be a C^1 vector field defined on all of $\mathbb{R}^n$, and let $x \in \mathbb{R}^n$. Let $\phi(t,x)$ be (unique) solution of the initial value problem.*

$$x' = X(x), \qquad x(0) = x.$$

By the Existence-Uniqueness Theorem for ordinary differential equations, such a solution exists on some open interval I containing 0. Assume that all such solutions can actually be defined for all real numbers t. Then, the function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a flow on $\mathbb{R}^n$.

(5) *Let (X, f) be a discrete dynamical system. Take a function $\tau : X \to \mathbb{R}$ such that $\tau(x) > 0$ for any $x \in X$. Define $Y = X \times \mathbb{R}/\sim$, where $(x, s + \tau(x)) \sim (f(x), s)$. Then define a flow or semiflow given by*

$$\phi^t((x,s)) = (x, t+s).$$

Hence, $\phi^{\tau(x)}((x,0)) = (f(x), 0)$. ϕ is called a suspension flow *for f, while Y is called a* suspension manifold *of X. τ is sometimes called a* roof function.

1.3. **Group actions.** Recall that a *group* G is a pair $(G.\cdot)$ consisting of a set G and a binary operation $\cdot$ called the product (or sum in the commutative case) satisfying

(i) (*associativity*) $\cdot$ is associative: $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ for all $g_1, g_2, g_3 \in G$;

(ii) (*identity*) there is an element $e \in G$ such that $e \cdot g = g \cdot e = g$ for all $g \in G$;

(iii) (*inverse*) for each $g \in G$, there there is an element $h \in G$ such that $g \cdot h = e = h \cdot g$.

If $\cdot$ only satisfies (i) and (ii), then G is called a *semigroup.*

Let $(G, \cdot)$ be a semi-group and X be a set. An action of $(G, \cdot)$ on X is a map $\Phi : G \times X \to X$ such that

(i) $\Phi(g \cdot h, x) = \Phi(g, \Phi(h, x))$ for all $g, h \in G$ and $x \in X$.

(ii) $\Phi(e, x) = x$ for all $x \in X$.

An *orbit* of a point $x \in X$ of a group action (X, G) is given by $O(x) = \{g(x) : g \in G\}$. $O(x)$ is *periodic* if $O(x)$ is a finite set. So if G is a finite group, every orbit is periodic.

More formally, an action of $(G, \cdot)$ is a map $\sigma : G \to (X)$, where (X) is the set of all selfmaps on X, such that

(i) $\sigma(g \cdot h) = \sigma(g) \circ \sigma(h)$ for all $g, h \in G$, and
(ii) $\sigma(e) = \mathrm{id}_X$.

With the notation, for any $x \in X$, $O(x) = \{\sigma(g)(x) : g \in G\}$.

A group action $\sigma : G \to (X)$ is *faithful* if σ is an injective, that is, for $g, h \in G$, $g \neq h$ implies $\sigma(g) \neq \sigma(h)$.

Example. (6) *$G = \mathbb{Z}$ or $\mathbb{Z}_+$ and $\cdot$ is the usual addition of integers. If $\sigma(1) = T$, then $\sigma(n) = T^n$ for $n \in \mathbb{Z}$ or $\mathbb{Z}_+$. The group action gives a discrete dymanical system.*

(7) *$G = \mathbb{R}$ or $\mathbb{R}_+$ and $\cdot$ is the usual addition of real numbers. If $\sigma(t_0) = \phi(t_0, \cdot)$ for any $t_0 \neq 0$, then $\sigma(s) = \phi(s, \cdot)T^n$ for any $s = rt_0$, where $r \in \mathbb{Q}$. If moreover both $\phi(t, \cdot)$ and $\sigma(t)$ are continuous on t, then $\sigma(t) = \phi(t, \cdot)$ for any $t \in \mathbb{R}$ or $\mathbb{R}_+$. The group action gives a continous dymanical system.*

(8) *$G = \mathbb{Z}^2$ or $\mathbb{Z}^2_+$. Then G is a Abelian group or semigroup. Suppose $X = [0, 1)$ and G is generated by the actions $f(x) = 2x \pmod 1$, and $g(x) = 3x \pmod 1$. The action is sometimes called the $(\times 2, \times 3)$ map, which is faithful.*

(9) *Let G be a discrete Heisenberg group, that is, the group*

$$\mathcal{H} = \{< a, b, c >: ac = ca, bc = cb, ab = bac\}.$$

or, quivalently,

$$\mathcal{H} = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{Z} \right\}.$$

Let

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $ac = ca$, $bc = cb$, $ab = bac$ and for every $k \in \mathcal{H}$, there is a unique triple $(n_1, n_2, n_3) \in \mathbb{Z}^3$ such that $k = a^{n_1} b^{n_2} c^{n_3}$.

Take $X = \mathbb{R}^3$ or $\mathbb{T}^3$. It is clear that each of a, b, c induces a map on $\mathbb{R}^3$ or $\mathbb{T}^3$. Then we get a discrete Heisenberg group action on X. Heisenberg group action is the simplest nonabelian group action.