1. (8 points) Let $\{a_n\}_{n\geq 0}$ be given by the following recursion.

$$a_{n+2} = a_{n+1} + 4a_n, \quad a_0 = 1,\ a_1 = 2$$

Now extend the given sequence to all of $\mathbb{Z}$ in a natural way. *Carefully*, find $a_{-1}, a_{-2}, a_{-3},$ and a_{-4}.
Hint: $a_{-5} = -7/1024$

Solution:

Here are the first few terms in this sequence

$$\frac{1}{4}, \frac{3}{16}, \frac{1}{64}, \frac{11}{256}, -\frac{7}{1024}, \frac{51}{4096}, -\frac{79}{16384}, \frac{283}{65536}, -\frac{599}{262144}, \frac{1731}{1048576}$$

2. (12 points) This is a continuation of the previous problem which we restate here for convenience.

   Let $\{a_n\}_{n\geq 0}$ be given by the following recursion.

$$a_{n+2} = a_{n+1} + 4a_n, \quad a_0 = 1,\ a_1 = 2$$

   Now extend the given sequence to all of $\mathbb{Z}$ in a natural way.

   Now let $\overline{A}(x) = \sum_{n\geq 1} a_{-n}x^n$.

   (a) Using Wilf Rules (or similar), find the closed form of $\overline{A}(x)$.

   Solution:

   Let $b_n = a_{-n}$ and let $B(x) = \sum_{n\geq 0} b_n x^n$. Now the given recursion becomes

$$4b_{n+2} = b_n - b_{n+1}, \quad b_0 = a_0 = 1,\ b_1 = a_{-1} = 1/4$$

   Then, by the WIlf Rules, we have

$$\frac{4(B(x) - b_0 - b_1 x)}{x^2} = B(x) - \frac{B(x) - b_0}{x}$$

   Clearing denominators and substituting the initial conditions yields

$$4B(x) - 4 - x = x^2 B(x) - xB(x) + x$$

   so that

$$B(x) = \frac{4 + 2x}{1 + x - x^2}$$

   It follows that

$$\begin{aligned} \overline{A}(x) = \sum_{n\geq 1} a_{-n}x^n &= B(x) - b_0 \\ &= \frac{4+2x}{1+x-x^2} - 1 \\ &= \frac{x(x+1)}{4+x-x^2} \end{aligned}$$

   (b) Use Theorem 3.9.1 from Sagan's text to find the closed form of $\overline{A}(x)$ and compare with your results from part (a) above.

   Solution:

   Let $A(x) = \sum_{n\geq 0} a_n x^n$. We leave it as an exercise to show that

$$A(x) = \frac{x+1}{1 - x - 4x^2}$$

   So, according to the referenced Theorem,

$$\begin{aligned} \overline{A}(x) = -A(1/x) &= -\frac{1/x + 1}{1 - (1/x) - 4(1/x)^2} \\ &= \frac{x(x+1)}{4 + x - x^2} \end{aligned}$$

   in agreement with part (a).