1. (8 points) Let P_j be the poset of all positive divisors of $2 \cdot 3^j$ and call $x \leq y$ if $x \mid y$. Find a simple formula for $\mu_{P_j}(x, y)$. *Hint:* A Hasse diagram may help.

Solution:

In class we stated that $[x, y] \cong [1, y/x]$. It follows that

$$\mu_{P_j}(x, y) = \begin{cases} 1, & \text{if } y/x \in \{1, 6\} \\ -1, & \text{if } y/x \in \{2, 3\} \\ 0, & \text{otherwise.} \end{cases}$$

2. Let m and n be distinct positive integers and let D_m and D_n be the corresponding divisor lattices.

   (a) (8 points) Under what condition(s) is $D_m \times D_n \cong D_{mn}$? Justify your claim.

   Solution:

   We claim that $D_m \times D_n \cong D_{mn}$ whenever m and n are relatively prime, i.e., $\gcd(m, n) = 1$.

   Let $\phi : D_m \times D_n \to D_{mn}$ be defined by $\phi(r, s) = rs$. The map is clearly surjective for if $w \in D_{mn}$, then $w \mid mn$ and we can write $w = ab$ where $a \mid m$ and $b \mid n$, so that $w = \phi(a, b)$.
   Now suppose that $uv = \phi(u, v) = \phi(w, z) = wz$. Then $u \mid wz$, so either $u \mid w$ or $u \mid z$. Clearly, $u \mid w$ since $u \mid m$ and $\gcd(m, n) = 1$. Similarly, we can reverse the argument to show that $w \mid u$. It follows that $u = w$ and hence $v = z$, so that $(u, v) = (w, z)$. Thus, ϕ is an injection and hence, a bijection.

   We leave it as an easy exercise to show that if $(a, b) \leq (x, y)$, then $\phi(a, b) \leq \phi(x, y)$. *Note:* Since $D_m \times D_n$ is finite, it is only necessary to show that ϕ is order-preserving because of Problem 2 on Quiz 9.

   It now follows that $D_m \times D_n \cong D_{mn}$ whenever $\gcd(m, n) = 1$.

   (b) (4 points) Show by example that $D_m \times D_n \ncong D_{mn}$ when the above condition(s) are not met.

   Solution:

   For example, $D_2 \times D_2 \cong B_2$ and has 4 elements. On the other hand, D_4 is a chain with only 3 elements, so the two are not isomorphic.