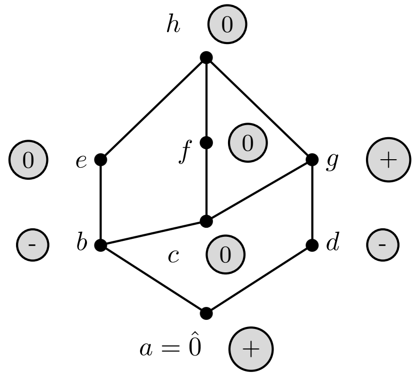

Figure 1: Poset P

1. The Hasse diagram of the poset P is shown in Figure 1. Answer the questions below.

   (a) (7 points) Find $\mu(\hat{0}, y)$ for all $y \in P$. Place these values next to the corresponding vertices in Fig. 1 as we have done in class.

   Solution:

   See Fig. 1. *Note:* In the figure, we set $- = -1$ and $+ = 1$.

   (b) (2 points) Find $\mu(b, h) = 1$.

   (c) (6 points) Use the linear extension $L : P \to [8]$, defined by $L(a) = 1, L(b) = 2, L(c) = 3, \ldots$ to construct the zeta matrix Z_L of P that corresponds to L.

   Solution:

$$Z_L = \begin{array}{c} \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

2. Let P be a finite poset and $f : P \to P$ be an order-preserving bijection.

   (a) (5 points) Show that f^{-1} is order preserving. *Note:* The statement is false if P is infinite.
   Hint: $f \circ f = f^2, f^3, f^4, \ldots$ are order-preserving.

   Solution:

   Since P is finite, $f^n = \mathrm{id}$ for some $n > 0$ (since f is a permutation). But then $\mathrm{id} = f \circ f^{n-1}$ so that $f^{-1} = f^{n-1}$ is order-preserving as we noted in the hint.

   (b) (5 points) Consider the following "proof" of part (a).

   > Suppose to the contrary that $a \leq b$ and that $f^{-1}(a) \geq f^{-1}(b)$. Then since f is order-preserving, $b = f(f^{-1}(b)) \leq f(f^{-1}(a)) = a$, contrary to our assumption.

   What is wrong with the above "proof"?

   Solution:

   The mistake is assuming that the contrary of $f^{-1}(a) \leq f^{-1}(b)$ is $f^{-1}(a) \geq f^{-1}(b)$. In fact, it is possible that $f^{-1}(a)$ and $f^{-1}(b)$ are incomparable.

   As an example, let $Q = \mathbb{Z} \cup \{s\}$ where $s < 0$ but s is incomparable to any negative integer. Now let $f : Q \to Q$ be defined by $f(s) = s$ and $f(n) = n + 1$ for $n \in \mathbb{Z}$ (see Fig. 2). It is easy to verify that f is an order-preserving bijection. Notice that $s < 0$, but $f^{-1}(s) = s$ and $f^{-1}(0) = -1$ are not comparable.

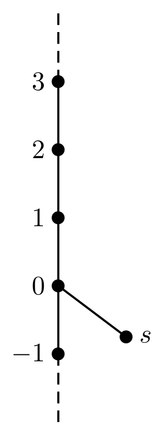

Figure 2: Poset Q