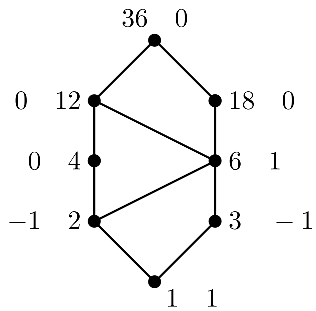

Figure 1: Poset P

1. Let $P = \{1, 2, 3, 4, 6, 12, 18, 36\}$ with (partial) order $\leq$ given by $x \leq y$ if $x \mid y$. Answer the questions below. *Note:* $P \neq D_{36}$ since $9 \notin P$.

   (a) (4 points) Carefully sketch the Hasse diagram of P.

   See Figure 1. Although it wasn't requested, Figure 1 also includes the Möbius values $\mu(1, y)$ for all $y \in P$. Can you find $\mu(2, y)$ for all $y \in P \setminus \{1, 3\}$.

   (b) (4 points) Find at least two pairs of elements in P that are not comparable (to each other).

   Solution:

   Two and 3 are not comparable nor are 4 and 18. Other answers are possible.

2. (12 points) Let $M(x)$ be ordinary generating function for the Motzkin numbers. In exercise 03/14.1 we saw that $M(x)$ satisfied the function equation

$$M(x) = 1 + xM(x) + x^2M(x)^2 \tag{1}$$

Use (1) to show that

$$[x^n]M(x) = \frac{1}{n+1}\sum_k \binom{n+1}{k}\binom{n+1-k}{k-1} \tag{2}$$

Hint: Multiple (1) by x and let $\overline{M}(x) = xM(x)$. Now use LIF to find a sum formula for $[x^n]\overline{M}(x)$.

Solution:

Let $\phi(z) = 1 + x + x^2$. Following the hint, we multiple (1) by x to obtain

$$xM(x) = x(1 + xM(x) + (xM(x))^2$$

or

$$\overline{M}(x) = x\phi(\overline{M}(x))$$

Lagrange inversion yields

$$\begin{aligned}
[x^n]\overline{M}(x) &= \frac{1}{n}[z^{n-1}](1+z+z^2)^n \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} z^{n-k}(1+z)^{n-k} \\
&= \frac{1}{n}[z^{n-1}]\sum_k\sum_j \binom{n}{k}\binom{n-k}{j} z^{n-k+j} \\
&= \frac{1}{n}\sum_k \binom{n}{k}\binom{n-k}{k-1}
\end{aligned}$$

But

$$\begin{aligned}
[x^n]M(x) &= [x^{n+1}]xM(x) = [x^{n+1}]\overline{M}(x) \\
&= \frac{1}{n+1}\sum_k \binom{n+1}{k}\binom{n+1-k}{k-1}
\end{aligned}$$

as desired.