1. For $n \geq 0$ show
$$[x^n](1+x+x^2)^n = [x^n]\frac{1}{\sqrt{1-2x-3x^2}} \tag{1}$$

   Hint: Use (3) below.

Solution:

Let $\phi(z) = 1 + z + z^2$ and let $z = z(x)$ be the unique power series satisfying $z = x\phi(z)$. Then

$$z = x(1+z+z^2)$$

which yields the solutions

$$z(x) = \frac{1 - x \pm \sqrt{1-2x-3x^2}}{2x}$$

and we may discard solution that blows up at $x = 0$ to obtain

$$z(x) = \frac{1 - x - \sqrt{1-2x-3x^2}}{2x}$$

It then follows by the right-hand side of (3) below that

$$\begin{aligned}
[z^n]\phi(z)^n &= [x^n]\frac{1}{1 - x\left\{1 + 2\left(\dfrac{1-x-\sqrt{1-2x-3x^2}}{2x}\right)\right\}} \\
&\vdots \\
&= [x^n]\frac{1}{\sqrt{1-2x-3x^2}}
\end{aligned}$$

which is (1).

2. Suppose that $z = z(x)$ satisfies the functional equation

$$z = x\phi(z) \tag{2}$$

Then for $n \geq 0$, show that

$$[z^n]\phi(z)^n = [x^n]\left\{\frac{xz'(x)}{z(x)}\right\} = [x^n]\frac{1}{1 - x\phi'(z(x))} \tag{3}$$

Solution:

For $n > 0$ we have

$$\begin{aligned}
[z^n]\phi(z)^n &\overset{(*)}{=} [z^{n-1}]\frac{1}{z}\,\phi(z)^n \\
&\overset{(**)}{=} n[x^n]\ln z(x) \\
&\overset{(*)}{=} [x^n]xD_x(\ln z(x)) \\
&= [x^n]\frac{xz'(x)}{z(x)}
\end{aligned}$$

Notice that we invoked a Wilf rule at both steps marked (*) and the (backwards) Lagrange Inversion formula at step (**). For the right-hand equality in (3), first notice that the functional equation (2) implies

$$z'(x) = \phi(z(x)) + x\phi'(z(x))z'(x)$$

Rearranging and solving for $z'(x)$ yields

$$z'(x) = \frac{\phi(z(x))}{1 - x\phi'(z(x))}$$

It now follows that

$$\begin{aligned}
[x^n]\frac{xz'(x)}{z(x)} &\overset{(\#)}{=} [x^n]\frac{1}{\phi(z(x))}\,z'(x) \\
&= [x^n]\frac{1}{\phi(z(x))}\,\frac{\phi(z(x))}{1 - x\phi'(z(x))} \\
&= [x^n]\frac{1}{1 - x\phi'(z(x))}
\end{aligned}$$

Here we utilized the functional equation (2) at step (#).

We leave the proof of (3) when $n = 0$ as an exercise.