1. (10 points) Let $\mathcal{T} = \mathcal{T}^{\Omega}$, where $\Omega = \{0, 2, 3\}$. So $\mathcal{T}$ is the Ω-restricted class of plane trees such that each node has either 0, 2, or 3 children. (See Definition 4 and Example 6 here.) The first few terms in the counting sequence for this class are 0, 1, 0, 1, 1, 2, 5, 8, 21, 42, 96, 222.

   Note: The size of each tree in $\mathcal{T}$ is measured by the number of vertices.

   (a) Sketch the two trees of size 5 and the five trees of size 6.

   (b) As usual, let $T(x)$ be the ordinary generating function for $\mathcal{T}$. Find the sum formula for $[x^n]T(x)$.
   Hint: What is the characteristic function for this class?

   Solution:

   Notice that $T(x)$ satisfies $T(x) = x\phi(T(x))$, with characteristic function $\phi(z) = 1 + z^2 + z^3$. So by the Lagrange Inversion formula,

$$
\begin{aligned}
[x^n]T(x) &= \frac{1}{n}[z^{n-1}](1+z^2+z^3)^n \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} z^{2n-2k}(1+z)^{n-k} \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} \sum_j \binom{n-k}{j} z^{2n-2k+j} \\
&= \frac{1}{n}\sum_k \binom{n}{k}\binom{n-k}{2k-n-1}
\end{aligned}
$$

   One of the reasons that we include the first few terms in these sequence questions is so that students can check to see if their solution actually works for early terms in the sequence. For example,

$$
\begin{aligned}
[x^5]T(x) &= \frac{1}{5}\sum_k \binom{5}{k}\binom{5-k}{2k-5-1} \\
&= \frac{1}{5}\left(0+0+0+\binom{5}{3}\binom{5-3}{2(3)-6}+0+0\right) \\
&= \frac{10}{5}
\end{aligned}
$$

   as expected.

2. (10 points) Let $\overline{\mathcal{T}} = \overline{\mathcal{T}}^{\Omega}$ where $\Omega = \{0, 2, 3\}$. However, this time we measure the size of each tree by the number of non-leaf vertices. Let $\overline{T}(x)$ be the ordinary generating function for $\overline{\mathcal{T}}$ and let $z(x) = \overline{T}(x) - 1$. One can show that $z = x\phi(z)$ for some characteristic function $\phi(z)$. Find $\phi(z)$.

   Hint: Such a tree is either $\circ$ (a vertex of size zero since it has no children) or $\mathcal{Z}_\bullet \times \overline{\mathcal{T}} \times \overline{\mathcal{T}}$ or $\mathcal{Z}_\bullet \times \overline{\mathcal{T}} \times \overline{\mathcal{T}} \times \overline{\mathcal{T}}$. Turn this into a class recursion and, once the recursion is established, take a look at Example 2.

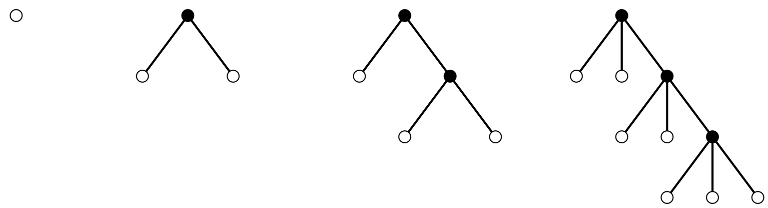

Figure 1: Trees of size 0, 1, 2, and 3.

Solution:

Figure 1 displays a few such trees. Nodes of weight zero are indicated using the symbol $\circ$. It turns out that $\phi(z) = (1+z)^2(2+z)$ so that

$$\begin{aligned}
[x^n]T(x) &= \frac{1}{n}[z^{n-1}](1+z)^{2n}(2+z)^n \\
&= \frac{1}{n}[z^{n-1}](2+5z+4z^2+z^3)^n \\
&= \frac{1}{n}\sum_{k}^{n}\binom{n}{k}\sum_{j}^{n-k}\binom{n-k}{j}\binom{n-k-j}{2n-3k-2j+1}2^k 5^j 4^{2n-3k-2j+1} \qquad (1)
\end{aligned}$$

The first few terms of this sequence are

$$1, 2, 10, 66, 498, 4066, 34970, 312066, 2862562, 26824386, 255680170, \ldots$$

Here are the details of the derivation of the characteristic function ϕ. According to the hint,

$$\overline{\mathcal{T}} = \mathcal{Z}_\circ + \mathcal{Z}_\bullet \times \overline{\mathcal{T}} \times \overline{\mathcal{T}} + \mathcal{Z}_\bullet \times \overline{\mathcal{T}} \times \overline{\mathcal{T}} \times \overline{\mathcal{T}} \qquad (2)$$

Notice that the ordinary generating functions of $\mathcal{Z}_\circ$ and $\mathcal{Z}_\bullet$ are x^0 and x^1, respectively. It follows that

$$\overline{T}(x) = 1 + x\overline{T}(x)^2 + x\overline{T}(x)^3 \qquad (3)$$

Now let $z(x) = \overline{T}(x) - 1$, then

$$\begin{aligned}
z(x) &= x((1+z(x)^2) + (1+z(x)^3)) \\
&= x(2+5z(x)+4z(x)^2+z(x)^3)
\end{aligned}$$

It follows that

$$\phi(z) = (2+5z+4z^2+z^3) \qquad (4)$$

and we are done.

Solution:

Here's the derivation of the formula (1) given above. Let $W(z) = 1 + z$. Then $W'(z) = 1$ and by the Lagrange Inversion formula

$$\begin{aligned}
[x^n]T(x) = [x^n]W(z(x)) &= \frac{1}{n}[z^{n-1}]W'(z)\phi(z)^n \\
&= \frac{1}{n}[z^{n-1}](2+5z+4z^2+z^3)^n \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} 2^k z^{n-k}(5+4z+z^2)^{n-k} \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} 2^k z^{n-k} \sum_j \binom{n-k}{j} 5^j z^{n-k-j}(4+z)^{n-k-j} \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} 2^k z^{n-k} \sum_j \binom{n-k}{j} 5^j z^{n-k-j} \sum_l 4^l \binom{n-k-j}{l} z^{n-k-j-l} \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} \sum_j \binom{n-k}{j} \sum_l \binom{n-k-j}{l} 2^k 5^j 4^l z^{n-k} z^{n-k-j} z^{n-k-j-l} \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} \sum_j \binom{n-k}{j} \sum_l \binom{n-k-j}{l} 2^k 5^j 4^l z^{3n-3k-2j-l}
\end{aligned}$$

Now $3n - 3k - 2j - l = n - 1$ implies that $l = 2n - 3k - 2j + 1$, so that

$$[x^n]T(x) = \frac{1}{n}\sum_k \binom{n}{k} \sum_j \binom{n-k}{j} \binom{n-k-j}{2n-3k-2j+1} 2^k 5^j 4^{2n-3k-2j+1}$$

as desired.