1. (8 points) Let $F(x) \overset{\text{ogf}}{\longleftrightarrow} \{f_n\}_n$ and $G(x) \overset{\text{ogf}}{\longleftrightarrow} \{g_n\}_n$. Suppose that $g_n = \sum_k \binom{n}{k} f_k$. Find a closed formula for $G(x) = \sum_{n\geq 0} g_n\, x^n$ in terms of $F(x)$. (C.f., example from Friday's class.)

**Solution:**

In class we showed that $g_n = \sum_k \binom{n}{k} f_k$ implies that

$$G(x) = \frac{1}{1-x} F\left(\frac{x}{1-x}\right)$$

2. (12 points) Let $\left\lfloor {[n] \atop k} \right\rfloor$ be the collection of all set partitions of $[n]$ with exactly $k$ blocks such that the elements within each block are ordered, and let $\left\lfloor {n \atop k} \right\rfloor = \left| \left\lfloor {[n] \atop k} \right\rfloor \right|$. Find the exponential generating function $F_{\left\lfloor {\cdot \atop k} \right\rfloor}(x)$ for $k \in \mathbb{P}$.

**Solution:**

$\left\lfloor {n \atop k} \right\rfloor$ are called the Lah numbers. First let $k = 1$. Since the block cannot be empty, it's easy to see that exponential generating function must be

$$F_{\left\lfloor {\cdot \atop 1} \right\rfloor}(x) = \frac{x}{1-x}$$

Why?

Now let $\left\lfloor {[n] \atop k} \right\rfloor_{\text{o}}$ denote the collection of all set partitions such that both the blocks and the elements within each block are ordered. Then by the product rule,

$$\left\lfloor {\cdot \atop k} \right\rfloor_{\text{o}} = \underbrace{\left\lfloor {\cdot \atop 1} \right\rfloor \times \left\lfloor {\cdot \atop 1} \right\rfloor \times \cdots \times \left\lfloor {\cdot \atop 1} \right\rfloor}_{k \text{ factors}}$$

It follows that the exponential generating for $\left\lfloor {\cdot \atop k} \right\rfloor_{\text{o}}$ is

$$F_{\left\lfloor {\cdot \atop k} \right\rfloor_{\text{o}}}(x) = \left(F_{\left\lfloor {\cdot \atop 1} \right\rfloor}\right)^k = \left(\frac{x}{1-x}\right)^k$$

and since the block order is irrelevant, we have

$$F_{\left\lfloor {\cdot \atop k} \right\rfloor}(x) = \frac{1}{k!}\left(\frac{x}{1-x}\right)^k$$