1. (10 points) Let $\mathcal{S}(\cdot) = 2^{\cdot}$ and $\mathcal{T}(\cdot) = \left\{ {\cdot \atop 2} \right\}$. Let $s_n = |2^{[n]}| = 2^n$ and let $t_n = |\left\{ {[n] \atop 2} \right\}| = \left\{ {n \atop 2} \right\}$. According to a theorem that we discussed in class today,

$$|(\mathcal{S} \times \mathcal{T})([3])| = \sum_{k=0}^{3} \binom{3}{k} s_k t_{3-k} = \binom{3}{0} \cdot 1 \cdot 3 + \binom{3}{1} \cdot 2 \cdot 1 + 0 + 0 = 9 \tag{1}$$

List all 9 elements in $(\mathcal{S} \times \mathcal{T})([3])$.

Solution:

Looking at (1), we should find 3 ordered pairs of the form $(\emptyset, \cdot/\cdot)$ where $\cdot/\cdot \in \left\{ {[3] \atop 2} \right\}$. We should also have 6 ordered pairs of the form $(\emptyset|k, \cdot/\cdot)$. Here $a|b$ means either a or b and $\cdot/\cdot \in \left\{ {L \atop 2} \right\}$ with $L \subset [3]$ and $|L| = 2$. Thus

$$(\mathcal{S} \times \mathcal{T})([3]) = \underbrace{\{(\emptyset, 1/23), (\emptyset, 12/3), (\emptyset, 13/2)\}}_{\text{first form}} \cup \underbrace{\{(\emptyset, 2/3), (1, 2/3), (\emptyset, 1/3), (2, 1/3), (\emptyset, 1/2), (3, 1/2)\}}_{\text{second form}}$$

We can actually say a bit more about this example. According to the product rule for exponential generating functions, we must have

$$\begin{aligned} F_{\mathcal{S}\times\mathcal{T}}(x) &= F_{\mathcal{S}}(x) \cdot F_{\mathcal{T}}(x) \\ &= e^{2x} \cdot \frac{(e^x - 1)^2}{2!} \qquad \text{(See exercise 02/07 Problem 1)} \\ &= \frac{1}{2}(e^{4x} - 2e^{3x} + e^{2x}) \end{aligned}$$

It follows that

$$\begin{aligned} 3![x^3]F_{\mathcal{S}\times\mathcal{T}}(x) &= \frac{6}{2}[x^3](e^{4x} - 2e^{3x} + e^{2x}) \\ &= 3\left(\frac{4^3 - 2 \cdot 3^3 + 2^3}{3!}\right) \\ &= 9 \end{aligned}$$

in agreement with (1). The first 12 terms in the counting sequence of the exponential generating function $F_{\mathcal{S}\times\mathcal{T}}(x)$ are

$$0, 0, 1, 9, 55, 285, 1351, 6069, 26335, 111645, 465751, 1921029, \ldots$$

2. (10 points) Let $\mathcal{S}(\cdot) = 2^{\cdot}$ and $\mathcal{T}(\cdot) = \left[{\cdot \atop 2}\right]$. List all elements in $(\mathcal{S} \times \mathcal{T})([4])$.

Solution:

From Table 4.3.1 in AoC, we have $F_{\mathcal{S}} = E^{2x}$ and $F_{\mathcal{T}} = \dfrac{\ln^2(1-x)}{2}$. Now, with some help from a CAS, we have

$$\begin{aligned} |(\mathcal{S} \times \mathcal{T})([4])| &= [x^4]F_{\mathcal{S}\times\mathcal{T}} \\ &= [x^4]e^{2x}\frac{\ln^2(1-x)}{2} \\ &= 59 \end{aligned}$$

It might be more instructive to mimic what we did in equation (1). So, let $s_n = |2^{[n]}| = 2^n$ and let $t_n = |\left[{[n] \atop 2}\right]| = \left[{n \atop 2}\right]$. Then

$$\begin{aligned} |(\mathcal{S} \times \mathcal{T})([4])| &= \sum_k \binom{4}{k} s_k\, t_{4-k} \\ &= \binom{4}{0}2^0\begin{bmatrix}4\\2\end{bmatrix} + \binom{4}{1}2^1\begin{bmatrix}3\\2\end{bmatrix} + \binom{4}{2}2^2\begin{bmatrix}2\\2\end{bmatrix} + \binom{4}{3}2^3\begin{bmatrix}1\\2\end{bmatrix} + \binom{4}{4}2^4\begin{bmatrix}0\\2\end{bmatrix} \\ &= 11 + 4\cdot 2\cdot 3 + 6\cdot 4\cdot 1 + 0 + 0 \\ &= 59 \end{aligned}$$

The advantage of this second approach is that we now know how the objects are constructed. So there are 11 objects of the form $(\emptyset, \pi)$ for some $\pi \in \left[{[4] \atop 2}\right]$. For example, $(\emptyset, (132)(4))$.

There 6 objects that have the form $(\emptyset|1, \pi)$ for some $\pi \in \left[{\{2,3,4\} \atop 2}\right]$. For example, $(\emptyset, (24)(3))$ and $(1, (24)(3))$. Six more that have the form $(\emptyset|2, \pi)$ for some $\pi \in \left[{\{1,3,4\} \atop 2}\right]$, and so on, for a total of 24. Here $a|b$ means either a or b.

Finally, there are 6 objects of the form $(s, (3)(4))$ where $s \in 2^{\{1,2\}}$. For example, $(\{1,2\}, (3)(4))$ and $(\{2\}, (3)(4))$. There are 6 more objects of the form $(s, (2)(4))$ where $s \in 2^{\{1,3\}}$, and so on, for a total of 24.

Putting this all together yields 59 objects, as expected.