1. (10 points) A coach wishes to break up her n-member team into 3 practice squads. Each player on squad A will wear either a red or a blue jersey, those on squad B will wear yellow jerseys numbered from 1 to |squad B|, and squad C players will wear black jerseys and choose a squad captain. Let $t_0 = 0$ and for $n > 0$, let t_n count the number of ways that she can do this. Find the closed form of the exponential generating function $\sum_n t_n\, x^n/n!$. *Note:* This means that the squad B team is ordered and that squad C must have at least one player.

   Hint: The first few terms in this sequence are $0, 1, 8, 51, 312, \ldots$

Solution:

Let $i, j,$ and k be the number of players resp. on squad A, squad B, and squad C. Then

$$t_n = \sum_{i+j+k=n} \frac{n!}{i!j!k!} 2^i\, j!\, k \tag{1}$$

So by the Wilf rules, we must have

$$\begin{aligned} T(x) = \sum_n t_n \frac{x^n}{n!} &= \sum_n 2^n \frac{x^n}{n!} \sum_n n! \frac{x^n}{n!} \sum_n n \frac{x^n}{n!} \\ &= e^{2x} \frac{1}{1-x}\, x e^x = \frac{x e^{3x}}{1-x} \end{aligned}$$

Although it wasn't requested, we can see that

$$t_n = n![x^n] \frac{x e^{3x}}{1-x} = n![x^{n-1}] \frac{1}{1-x} e^{3x} = n! \sum_{k=0}^{n-1} \frac{3^k}{k!}$$

So, for example,

$$t_3 = 3! \left(1 + \frac{3}{1} + \frac{3^2}{2!}\right) = 6(1 + 3 + 9/2) = 51, \text{ as expected}$$

It is also worthwhile to list the 8 possibilities with two players, say 1 and 2. For example, $1^r|0|2$ indicates that player 1 is given a red jersey on squad A and player 2 would be the captain of squad C. Here are the other 7 lineups.

$$1^b|0|2,\ 2^r|0|1,\ \ 2^b|0|1$$
$$0|1|2,\ 0|2|1$$
$$0|0|1^c2,\ 0|0|12^c$$

Hopefully, the notation is self-explanatory and it should be clear that these are the only cases.

Remark: (a) Notice that the right-hand side of (1) is zero whenever $k = 0$. That is, for any configuration that assigns zero players to squad C, the summand is 0, as expected.

(b) It is a worthwhile exercise to list the 51 possibilities for 3 players. This is something I plan to advocate for the entire semester - *List all possible configurations (within reason) for any exercise that you question.* Also, don't be afraid to use https://oeis.org/.

2. (a) (6 points) Use exercise 01/20 - #2 to show

$$\sum_{n,k\geq 0} \left\{ {n \atop k} \right\} t^k \frac{x^n}{n!} = e^{t(e^x-1)} \tag{2}$$

Solution:

Following the suggestion, we have

$$\begin{aligned}
\sum_{n\geq 0} \left\{ {n \atop k} \right\} \frac{x^n}{n!} &= \frac{1}{k!} \sum_{n\geq 0} k! \left\{ {n \atop k} \right\} \frac{x^n}{n!} \\
&= \frac{1}{k!} \sum_{n\geq 0} \sum_{j=0}^{k} \binom{k}{j} j^n (-1)^{j+k} \frac{x^n}{n!} \\
&= \frac{(-1)^k}{k!} \sum_{j} (-1)^j \binom{k}{j} \sum_{n\geq 0} \frac{(jx)^n}{n!} \\
&= \frac{(-1)^k}{k!} \sum_{j} (-1)^j \binom{k}{j} e^{xj} \\
&= \frac{(-1)^k}{k!} \sum_{j} \binom{k}{j} (-e^x)^j \\
&= \frac{(-1)^k}{k!} (1-e^x)^k \quad \text{(by the Binomial Theorem)}
\end{aligned}$$

We give an another proof of this important identity in the remarks that follow part (b). Now

$$\begin{aligned}
\sum_{n,k\geq 0} \left\{ {n \atop k} \right\} t^k \frac{x^n}{n!} &= \sum_{k\geq 0} t^k \sum_{n\geq 0} \left\{ {n \atop k} \right\} \frac{x^n}{n!} \\
&= \sum_{k\geq 0} \frac{t^k (e^x-1)^k}{k!} = e^{t(e^x-1)}
\end{aligned}$$

as desired.

(b) (4 points) Let b_n be the Bell numbers, $b_n = \sum_k \left\{ {n \atop k} \right\}$. Use (2) to show that

$$\sum_{n\geq 0} b_n \frac{x^n}{n!} = e^{e^x-1}$$

No credit for any other method.

Solution:

We have

$$\sum_{n\geq 0} b_n \frac{x^n}{n!} = \sum_{n\geq 0} \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \sum_{n\geq 0} \sum_{k=0}^{n} \left\{ {n \atop k} \right\} 1^k \frac{x^n}{n!} = e^{1(e^x-1)}$$

Remark: Let $F_k(x) = \sum_{n\geq 0} \left\{ {n \atop k} \right\} \frac{x^n}{n!}$. Use induction on k to show that

$$F_k(x) = \frac{(e^x - 1)^k}{k!} \tag{3}$$

Proof: It is easy to see that $F_0(x) = 1$ in agreement with (3). Now suppose that (3) holds for all $j < k$. Then

$$\begin{aligned}
F_k'(x) &= \sum_{n\geq 0} \left\{ {n \atop k} \right\} n \frac{x^{n-1}}{n!} \\
&= \sum_{n\geq 1} \left\{ {n \atop k} \right\} \frac{x^{n-1}}{(n-1)!} \\
&\overset{(*)}{=} \sum_{n\geq 1} \left\{ {n-1 \atop k-1} \right\} \frac{x^{n-1}}{(n-1)!} + \sum_{n\geq 1} k \left\{ {n-1 \atop k} \right\} \frac{x^{n-1}}{(n-1)!} \\
&= \sum_{n\geq 0} \left\{ {n \atop k-1} \right\} \frac{x^n}{n!} + k \sum_{n\geq 0} \left\{ {n \atop k} \right\} \frac{x^n}{n!} \\
&= F_{k-1}(x) + kF_k(x)
\end{aligned}$$

Rearranging and using the induction hypothesis, we obtain

$$F_k'(x) - kF_k(x) = \frac{(e^x - 1)^{k-1}}{(k-1)!}$$

Now we mulitiply by the intergrating factor e^{-kx} to obtain

$$e^{-kx} F_k'(x) - ke^{-kx} F_k(x) = \frac{e^{-x}}{(k-1)!} \left(1 - e^{-x}\right)^{k-1}$$

or

$$D\left(e^{-kx} F_k(x)\right) = \frac{e^{-x}}{(k-1)!} \left(1 - e^{-x}\right)^{k-1}$$

Integrating both sides produces

$$e^{-kx} F_k(x) = \frac{(1 - e^{-x})^k}{k!} + C = \frac{(1 - e^{-x})^k}{k!} + 0$$

which is equivalent to (3). Notice that we used the following recursion at step (*).

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$ □