We begin with a standard bit of notation. If P is a poset with a $\hat{0}$ and $\hat{1}$, then

$$\mu(P) = \mu_P(\hat{0}, \hat{1}) = \mu(\hat{0}, \hat{1})$$

Let's recap a quick way to compute Möbius function values on some important posets.

a. If $x < y \in [n]$ then $\mu(x, y) = -1$ if $x + 1 = y$ and $\mu(x, y) = 0$ otherwise.

b. If C_n is a chain, then

$$\mu(C_n) = \mu(\hat{0}, \hat{1}) = \begin{cases} 1 & \text{if } n = 0, \\ -1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

c. If $S \leq T \in B_n$ (the Boolean algebra), then $\mu(S, T) = (-1)^{|T-S|}$.

d. If $x < y \in D_n$ (the Divisor lattice), then

$$\mu(x, y) = \begin{cases} (-1)^k & \text{if } y/x \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

Remark. Notice that (a) is an immediate consequence of (b).

Computing Möbius function values on set partitions

It turns out that we can quickly compute Möbius function values on set partitions, but first we need to prove a preliminary result. We have the following

Lemma 1. Let $x < y \in \Pi_n$, the lattice of set partitions under the usual refinement order. Now suppose that $y = B_1/B_2/\cdots/B_k$ and that each of blocks B_i is further partitioned into n_i blocks in x, then

$$[x, y] \cong \prod_{i=1}^{k} \Pi_{n_i} \tag{1}$$

In particular,

$$\mu(x, y) = \prod_{i=1}^{k} \mu(\Pi_{n_i}), \tag{2}$$

Notice that (2) is an immediate consequence of Theorem 16.24. We temporarily postpone the proof of (1).

Now because of Lemma 1, it appears that we need a quick way to compute $\mu(\Pi_n)$ for any $n > 0$. We have already manually computed $\mu(\Pi_1) = 1$, $\mu(\Pi_2) = -1$ and $\mu(\Pi_3) = 2$. And it's not difficult to discover that $\mu(\Pi_4) = -6$. We have the following

Theorem 2. Let Π_n be the lattice of set partitions under the usual refinement order. Then

$$\mu(\Pi_n) = (n-1)!\,(-1)^{n-1} \tag{3}$$

Proof: Let $x \in \Pi_n$ be a set partition and let $|x|$ denote the number of blocks of x. Using $q \geq n$ colors, in how many ways can we assign colors to the integers of $[n]$ so that members of the same block of x share the same color? Evidently, there are $q^{|x|}$ ways to do this. Call this a *block coloring*. Now suppose that we have such a coloring of the blocks of x. We can now create a new partition y in Π_n by combining all of the blocks of x that have same color. Such a partition would then have blocks with distinct colors. Call this a *distinct block coloring*. Notice that there are

$$(q)_{|y|} = q(q-1)\cdots(q-|y|+1)$$

ways to do this.

It follows that

$$q^{|x|} = \sum_{y \geq x} (q)_{|y|}$$

In words, the last line states that the number of block colorings of x is the sum of the number of all distinct block coverings of y for $y \geq x$. So by Möbius inversion,

$$(q)_{|x|} = \sum_{y \geq x} q^{|y|} \mu(x, y)$$

Now let $x = \hat{0}$ to produce

$$(q)_{|\hat{0}|} = (q)_n = \sum_{y \geq \hat{0}} q^{|y|} \mu(\hat{0}, y)$$

Now observe that

$$[q^1] \sum_{y \geq \hat{0}} q^{|y|} \mu(\hat{0}, y) = \mu(\hat{0}, \hat{1})$$

It follows that

$$\begin{aligned} \mu(\hat{0}, \hat{1}) &= [q^1](q)_n \\ &= [q^0](q-1)(q-2)\cdots(q-(n-1)) \\ &= (-1)^{n-1}(n-1)! \end{aligned}$$

as expected. □

Can we drop the requirement that $q \geq n$?