The Lagrange Inversion Formula (cont)

We restate the Lagrange Inversion Theorem here for convenience. Once again, suppose we have the following functional equation.

$$z = x\phi(z) \tag{1}$$

Can we solve for z as an explicit function of x? Can we find a closed formula for the sequence of coefficients, $[x^n]z(x)$? *Note:* The functional equation (1) implies $z(0) = 0$.

Theorem 1 (The Lagrange inversion formula (LIF)). Suppose that $W(z)$ and $\phi(z)$ are formal power series in z with $\phi(0) \neq 0$. Then there is a unique formal power series $z = z(x) = \sum_n z_n x^n$, satisfying (1). In addition, the value of $W(z(x))$ when expanded in a power series in x about $x = 0$ satisfies

$$n[x^n]W(z(x)) = [z^{n-1}]\{W'(z)\phi(z)^n\} \tag{2}$$

In a previous lecture, we proved (2) using induction. Before introducing the second proof, we need to extend the ring of formal power series $\mathbb{C}[[x]]$ to the ring of formal Laurent series,

$$C((x)) = \left\{ \sum_{n \geq -N} c_n x^n \,\middle|\, N \in \mathbb{Z}, c_n \in \mathbb{C} \right\} \tag{3}$$

Notice that if $f(x) \in \mathbb{C}((x))$, then $\min\{n \,|\, [x^n]f(x) \neq 0, n \in \mathbb{Z}\} > -\infty$. Many of the rules from formal power series carry over directly to the ring of formal Laurent series, including sums, products, and the formal derivative. In addition, coefficient extraction works in $\mathbb{C}((x))$ just as it does in $\mathbb{C}[[x]]$. We also have the following

Lemma 2.

(i) If $z(x) \in \mathbb{C}((x))$ then $[x^{-1}]z'(x) = 0$.

(ii) If $f(x) \in \mathbb{C}[[x]]$ with $f(0) = 0$ and $f'(0) \neq 0$, then

$$[x^{-1}]f(x)^k f'(x) = \delta_{-1}(k) \tag{4}$$

for $k \in \mathbb{Z}$.

Proof: (i) is obvious. Let $f(x) = \sum_{n\geq 1} a_n x^n$, with $[x^1]f(x) = f'(0) \neq 0$. If $k = -1$ we have

$$\begin{aligned} f(x)^{-1}f'(x) &= \frac{a_1 + 2a_2x + \cdots}{a_1x + a_2x^2 + \cdots} \\ &= \frac{1}{x} + \frac{a_2}{a_1} + \cdots \end{aligned}$$

so that

$$[x^{-1}]f(x)^{-1}f'(x) = [x^{-1}]\frac{1}{x} + [x^{-1}]\left(\frac{a_2}{a_1} + \cdots\right)$$
$$= 1 + 0$$

If $k \neq -1$ then

$$[x^{-1}]f(x)^k f'(x) = [x^{-1}]\frac{1}{k+1}(f(x)^{k+1})' = 0$$

by (i) above □

Remark: Notice that if $k < 0$, then $f(x)^k \notin C[[x]]$ since $[x^0]f(x) = f(0) = 0$. See Proposition 2 on the handout about formal power series. However, $f(x)^k \in C((x))$, the ring of formal Laurent series. We have more to say about this in the exercises.

Our second proof is lifted from the volume 2 of R. Stanley's *Enumerative Combinatorics*. We will show that although Stanley's version appears to be different, it is equivalent to the version presented above. We have

Theorem 3 (The Lagrange inversion formula). Let $F(x) = \sum_{n\geq 1} f_n\, x^n$ be a formal power series with $f_1 \neq 0$. Then

$$n[x^n]\{F^{<-1>}(x)\}^k = k[x^{n-k}]\left(\frac{x}{F(x)}\right)^n, \quad k \in \mathbb{Z} \tag{5}$$

Here, $F^{<-1>}(x)$ is the compositional inverse of $F(x)$. That is, $F^{<-1>}(F(x)) = F(F^{<-1>}(x)) = x$.

Proof (Second proof of LIF): So let

$$\{F^{<-1>}(x)\}^k = \sum_{j\geq k} p_j\, x^j \tag{6}$$

Then

$$x^k = \left\{F^{<-1>}(F(x))\right\}^k = \sum_{j\geq k} p_j\, F(x)^j$$

After differentiating both sides and dividing by $F(x)^n$ we obtain

$$\frac{kx^{k-1}}{F(x)^n} = \sum_{j\geq k} j\, p_j\, F(x)^{j-n-1}F'(x) \tag{7}$$

Notice that we are treating both sides of (7) as Laurent series. For example,

$$\frac{kx^{k-1}}{F(x)^n} = \frac{kx^{k-1}}{(f_1x + f_2x^2 + \cdots)^n}$$
$$= kx^{k-n-1}(f_1 + f_2x + \cdots)^{-n}$$

And the last expression is an element of $\mathbb{C}((x))$ since $(f_1 + f_2 x + \cdots)^{-n} \in \mathbb{C}[[x]]$. Now by Lemma 2

$$[x^{-1}]F(x)^{j-n-1}F'(x) = \delta_{-1}(j-n-1) = \delta_n(j) \tag{8}$$

so that

$$\begin{aligned} k[x^{n-k}]\left(\frac{x}{F(x)}\right)^n &= [x^{-1}]\frac{kx^{k-1}}{F(x)^n} \\ &\overset{(7)}{=} [x^{-1}]\sum_{j\geq k} j\, p_j\, F(x)^{j-n-1}F'(x) \\ &\overset{(8)}{=} \sum_{j\geq k} j\, p_j\, \delta_n(j) \\ &= np_n \end{aligned}$$

Thus

$$k[x^{n-k}]\left(\frac{x}{F(x)}\right)^n = np_n = n[x^n]\sum_{j\geq k} p_j\, x^j \overset{(6)}{=} n[x^n]\{F^{<-1>}(x)\}^k$$

as desired. $\square$

To see that the two versions are equivalent, we let $z(x) = F^{<-1>}(x)$ and let $\phi(x) = x/F(x)$. Then $F(z(x)) = x$ and

$$x\phi(z(x)) = x\frac{z(x)}{F(z(x))} = x\frac{z(x)}{x}$$

so that (1) is satisfied. Making the appropriate substitutions in (5), we have

$$n[x^n]z(x)^k = k[x^{n-k}]\phi(x)^n = [x^{n-1}]kx^{k-1}\phi(x)^n \tag{9}$$

which is equivalent to our original version, except that (9) holds for any $k \in \mathbb{Z}$.

Example 4. In 1870, the German mathematician Ernst Schröder asked the following question. In how many ways can n identical variables be "bracketed"? We give a recursive definition: x is a bracketing. And for $k \geq 2$, if $\delta_1, \delta_2, \ldots, \delta_k$ are bracketed expressions, then so is $(\delta_1 \cdot \delta_2 \cdots \delta_k)$. For example, $x, (xx),$ and $(x(xx))$ are bracketed expressions and $(xxx), (x(xx)), ((xx)x)$ are the three bracketings of size 3. If $\mathcal{S}$ is the class of all bracketings, then

$$\mathcal{S} = \mathcal{Z} + \mathrm{SEQ}_{\geq 2}(\mathcal{S}) \quad \Longrightarrow \quad S(x) = x + \frac{S(x)^2}{1-S(x)} \tag{10}$$

Although the Lagrange Inversion formula does not directly apply to right-hand side of (10), one can solve the equation to conclude

$$S(x) = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4} \tag{11}$$

The counting sequence of $S(x)$ begins with $0, 1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, \ldots$

We note that right-hand side of (10) can be rearranged so that the Lagrange Inversion formula applies. We explore this in the exercises.

Exercises

1. Let $f(x) = \sum_{n\geq 1} f_n\, x^n \in x\,\mathbb{C}[[x]]$. For any $g(x) \in \mathbb{C}((x))$, define the degree of $g(x)$ as we did for formal power series. That is,

$$\deg(g(x)) = \min\{n \in \mathbb{Z} \,|\, [x^n]g(x) \neq 0\}$$

Let $k > 0$. Show that $f(x)^{-k} \in \mathbb{C}((x))$ with $\deg(f(x)^{-k}) = -k$. (Also, see Lemma 2.)

2. What happens in (6) if $k > n$? Is this consistent with (5)? *Hint:* Use polynomial division to write out the first few terms in $x/F(x)$.

3. In this problem, we investigate the Schröder bracketing problem from Example 4.

   (a) List the 11 bracketings of size 4.

   (b) Use (11) to show that

$$[x^n]S(x) = \frac{\delta_0(n) + \delta_1(n) + \sum_{k\geq 0} \binom{1/2}{k}\binom{k}{2k-n}(-6)^{2k-n}}{4}$$

   (c) Rearrange the defining equation for $S(x)$ given on the right-hand side of (10) so that the Lagrange Inversion formula can be applied. What is $\phi(z)$?

   (d) Now use the Lagrange Inversion formula to show that

$$[x^n]S(x) = \frac{1}{n}\sum_{k\geq 0}\binom{2n-k-2}{n-1}\binom{n-2}{k}$$

4. Now let $\mathcal{F}_k$ to the class of *k-ordered* forests defined by $\mathcal{F}_k = \mathrm{SEQ}_k(\mathcal{T})$ where $\mathcal{T}$ is a plane tree and, once again, we measure the size of a forest by the number of vertices. For example, in $\mathcal{F}_2$, there are zero forests of size 1, one forest of size 2, and two forests of size 3. For the last case, we have (•, •–•) and (•–•, •). *Note:* There is only one plane tree of size 2, so it is shown here as a barbell: •–•.

(a) List all of the 2-ordered forests of size 4. There should be five of them.

(b) Let $F_n^k = [x^n]F_k(x)$. According the Lagrange Inversion formula, $F_n^k = [x^n]T(x)^k = \frac{k}{n}[z^{n-k}]\phi(z)^n$. Find a closed formula for F_n^k.

(c) Notice that

$$F_n^k = [x^n]\left(\frac{1-\sqrt{1-4x}}{2}\right)^k \tag{12}$$

Use a CAS (such as MatLab or Wolfram Alpha) to verify (12) for $k \in \{2, 3\}$.

5. Let $\mathcal{F}_k = \mathrm{SEQ}_k(\mathcal{T}^\Omega)$. Repeat the part (b) of the previous exercise for each of the following Ω-restricted trees.

(a) $\Omega = \{0, 2\}$

(b) $\Omega = \{0, 1, 2\}$