Lecture 3 - Integer Partitions - Part 2 (Generating Functions)

Example 1. In how many ways can we make change for a dollar with the following collections of coins.

a. Using 5 pennies, 8 nickels, 3 dimes and 4 quarters.

b. Using an unlimited supply of pennies, nickels, dimes and quarters.

For part (a), it turns out to be easier to answer a more general question, "In how many ways can we make change for any monetary denomination using the specified collection of coins?"

For example, if we chose 2 pennies, 2 nickels, 1 dime, and 1 quarter, that would be one way to "make change" for 47 cents. That correspond to choosing x^2 from the first factor, x^{10} from the second factor, x^{10} from the third factor and x^{25} from the last factor of the expression

$$\left(1 + x + \cdots + x^5\right)\left(1 + x^5 + \cdots + x^{40}\right)\left(1 + x^{10} + x^{20} + x^{30}\right)\left(1 + x^{25} + \cdots + x^{100}\right)$$

With the help of a CAS, we can easily expand the previous expression to obtain

$$1 + x + x^2 + x^3 + x^4 + 2x^5 + \cdots + 7x^{25} + \cdots + 6x^{47} + \cdots + 14x^{100} + \cdots + 7x^{150} + \cdots + x^{175} \quad (1)$$

Notice that there are 6 ways to make change for 47 cents. Here is the complete list:

$$\begin{aligned} 47 &= 25 + 10 + 10 + 1 + 1 \\ &= 25 + 10 + 5 + 5 + 1 + 1 \\ &= 25 + 5 + 5 + 5 + 5 + 1 + 1 \\ &= 10 + 10 + 10 + 5 + 5 + 5 + 1 + 1 \\ &= 10 + 10 + 5 + 5 + 5 + 5 + 5 + 1 + 1 \\ &= 10 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 1 + 1 \end{aligned}$$

Notice that the above list includes exactly 6 of the 124,754 integer partitions of 47.

Scanning (1) we see that there are 7 ways to make change for a quarter and 14 ways to make change for a dollar.

What about part (b)? First, recall that if k is a positive integer, then

$$\sum_{n\geq 0} x^{k\cdot n} = \frac{1}{1 - x^k} \quad (2)$$

Now an unlimited supply of pennies and nickels could be represented by $\sum_n x^n$ and $\sum_n x^{5n}$ respectively. With the help of (2) and proceeding as we did in part (a), the answer should be $[x^{100}]C(x)$, where

$$\begin{aligned} C(x) &= \sum_{n\geq 0} x^n \sum_{n\geq 0} x^{5n} \sum_{n\geq 0} x^{10n} \sum_{n\geq 0} x^{25n} \\ &= \frac{1}{1-x}\,\frac{1}{1-x^5}\,\frac{1}{1-x^{10}}\,\frac{1}{1-x^{25}} \end{aligned}$$

Now, with the help of a computer, we can calculate that there are $[x^{100}]C(x) = 242$ ways to make change for a dollar with an unlimited supply of pennies, nickels, dimes and quarters.

Motivated by the previous example, we have the following

Theorem 2. (Euler) Let $p(n)$ be the number of ways to partition the nonnegative integer n. Then

$$\mathcal{E}(x) = \sum_{n\geq 0} p(n)x^n = \frac{1}{1-x}\,\frac{1}{1-x^2}\,\frac{1}{1-x^3}\,\cdots = \prod_{j\geq 1}(1-x^j)^{-1} \tag{3}$$

Remark. Since that right-hand side of (3) involves an infinite product, it does not appear to be a legitimate rational generating function since computing any coefficient would seem to require an infinite number of operations. However, this is misleading since

$$p(n) = [x^n]P(x) = [x^n]\frac{1}{1-x}\,\frac{1}{1-x^2}\,\cdots\,\frac{1}{1-x^n} \tag{4}$$

$$= [x^n]\left(\frac{q_1(x)}{1-x} + \frac{q_2(x)}{1-x^2} + \cdots + \frac{q_n(x)}{1-x^n}\right) \tag{5}$$

where $\deg(q_k) \leq k-1$.

Example 3. Let $f(n)$ be the number of partitions of n that have no part equal to 2. Then

$$\begin{aligned} F(x) = \sum_{n\geq 0} f(n)x^n &= \frac{1}{1-x}\cdot 1\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^4}\cdots \\ &= \frac{1}{1-x}\cdot\frac{1-x^2}{1-x^2}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^4}\cdots \\ &= (1-x^2)\mathcal{E}(x) \end{aligned}$$

It follows that

$$\begin{aligned} f(n) &= [x^n](1-x^2)\mathcal{E}(x) \\ &= [x^n]\mathcal{E}(x) - [x^{n-2}]\mathcal{E}(x) \\ &= p(n) - p(n-2), \quad n \geq 2 \end{aligned}$$

Example 4. Can you guess what the following generating functions might count?

$$\begin{aligned} d(x) &= \prod_{j\geq 1}(1+x^j) \\ O(x) &= \prod_{j\geq 0}(1-x^{2j+1})^{-1} \\ D(x) &= (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots \\ &= \prod_{j\geq 0}(1+x^{2^j}) \end{aligned}$$

Definition 5. Let $p_d(n)$ be the number of ways to partition n into distinct parts and let $p_o(n)$ be the number of ways to partition n into parts, all of which are odd. For example, two partitions of 10 are $5+5$ and $4+3+2+1$. The first is a partition of 10 into odd parts and the second is a partition of 10 into distinct parts.

Theorem 6. For $n \geq 0$,

$$p_d(n) = p_o(n) \tag{6}$$

Proof: In Example 4 we argued, in class, that $\sum_{n\geq 0} p_d(n)x^n = d(x)$ and $\sum_{n\geq 0} p_0(n)x^n = O(x)$. Hence it suffices to show that $d(x) = O(x)$.

$$\begin{aligned} d(x) &= (1+x)(1+x^2)(1+x^3)\cdots \\ &= \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^6}{1-x^3}\cdot\frac{1-x^8}{1-x^4}\cdots \\ &= \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}\cdots \\ &= O(x) \end{aligned}$$

□

Exercises

1. Let $D(x) = \prod_{j\geq 0}(1 + x^{2^j})$ from Example 4. Find a combinatorial proof that $D(x) = (1 - x)^{-1}$.
   Hint: Show that $[x^n]D(x) = 1$ for all n.

2. Let $g(n)$ count the number of partitions of n that have no part equal to 1 or 2. Express $g(n)$ in terms of $p(n)$.

3. There is another compelling (but not rigorous?) argument that computing $[x^n]\mathcal{E}(x)$ involves only a finite number of operations. Can you find it?
   Hint: The starting point is the right-hand side of (4).

4. Which of the following are formal power series? For those that are, use the extractionator to identify the nth coefficient of its power series expansion.

$$f(x) = \frac{1}{(1-3x)^2}$$

$$g(x) = \sum_{n\geq 0} \left(\!\binom{n}{5}\!\right) 3^{n-1}x^n$$

$$k(x) = \sum_{n\geq 0}(1+x)^n$$

$$l(x) = \sum_{n\geq 0}(x+x^2)^n$$

$$q(x) = \sum_{n\geq 0}\frac{1}{(1-x)^n}$$

$$r(x) = \sum_{n\geq 0}\frac{x^n}{(1-x)^n}$$

$$F(x) = \frac{1}{1-x-x^2}$$

   For example, $k(x)$ is **not** a formal power series since $k(x) = u(v(x))$ where $v(x) = 1 + x$. But $v(0) \neq 0$ contrary to the restrictions placed on compositions in Wilf.