Compositions

In Math 481 we defined and discussed multisets and various equivalent ways that certain problems were equivalent to counting multisets. One of these was the following:

Definition 1. Let n, k be integers. Then $\left(\!\binom{k}{n}\!\right)$ is the total number of nonnegative solutions to the equation

$$n = x_1 + x_2 + \cdots + x_k \tag{1}$$

The right-hand side of (1) is called a *weak composition* of n into k parts. If we insist that the $x_j > 0$, then (1) is called a *composition* of n into k parts.

For example, $4 + 0 + 12 + 10$ is a weak composition of 24 into 4 parts and $4 + 8 + 9 + 3$ is a composition of 24 into 4 parts. It's important to remember that order matters, so that $4 + 8 + 9 + 3$ and $8 + 4 + 9 + 3$ are different compositions of 24 into 4 parts.

We restate two of the results from Math 481 using the language of compositions.

Theorem 2. Let n, k be nonnegative integers. Then the number of weak compositions of n into k parts is

$$\left(\!\!\binom{k}{n}\!\!\right) = \binom{n+k-1}{n} = \binom{n+k-1}{k-1} \tag{2}$$

and the number of compositions of n into k parts is

$$\left(\!\!\binom{k}{k-n}\!\!\right) = \binom{n-1}{n-k} = \binom{n-1}{k-1} \tag{3}$$

The next result is new.

Theorem 3. Let n be a positive integer. Then the number of compositions of n is 2^{n-1}.

Proof: Let n be a positive integer. Then n can be written as a composition into 1 part, or 2 parts, etc. So by the Addition rule and Theorem 2, the number of compositions of n is

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{k} \binom{n-1}{k} = 2^{n-1}$$ □

The set of compositions of n into k parts is often denoted by the symbol $Q([n], k)$ and the set of all compositions of n is denoted by $Q([n])$. The size of these two sets is then denoted $q(n, k)$ and $q(n)$, respectively. With this notation, the last two results can be restated as

$$q(n, k) = |Q([n], k)| = \binom{n-1}{k-1}$$

$$q(n) = |Q([n])| = 2^{n-1}$$

Integer Partitions

In Math 481 we also introduced set partitions and the Bell numbers. It turns out that we can develop a similar concept with integers.

Definition 4. Let n be a nonnegative integer. An *integer partition* of n is a multiset λ whose elements (called *parts*) sum to n. We introduce the notation $\lambda \vdash n$ to mean that λ is an integer partition of n. Since the elements of a multiset set have no inherent order, we will always list the elements of λ as a weakly decreasing sequence. In keeping with earlier conventions, the set of all integer partitions of n will be denoted $P([n])$ and its size is given by $p(n)$. That is, $p(n) = |P([n])|$.

For example, $(3, 1, 1) \vdash 5$ and the set of all integer partitions of 5 is

$$P([5]) = \{(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)\}$$

It follows that $p(5) = 7$.

There is a useful visualization for integer partitions. A *Young diagram* (or *Ferrers shape*) of an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ is a left-justified array of squares whose jth row has λ_j squares. Figure 1 shows the Young diagram for $\lambda = (5, 3, 2, 2) \vdash 12$. It also includes its transpose or *conjugate*, λ^{t}.

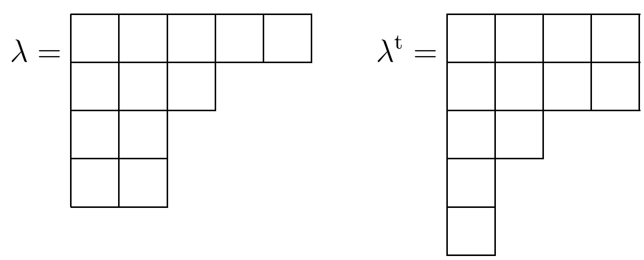

Figure 1: Young diagram and its transpose for the integer partition $\lambda = (5, 3, 2, 2)$

Notice that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ and $\lambda^{\mathrm{t}} = (\lambda^{\mathrm{t}}_1, \lambda^{\mathrm{t}}_2, \ldots, \lambda^{\mathrm{t}}_m)$ is its transpose, then λ^{t} is an integer partition of n whose jth part counts the number of parts of λ that are greater than or equal to j.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	0	1	1	1	1	1	1	1	1	1
2	0	1	2	2	2	2	2	2	2	2
3	0	1	2	3	3	3	3	3	3	3
4	0	1	3	4	5	5	5	5	5	5
5	0	1	3	5	6	7	7	7	7	7
6	0	1	4	7	9	10	11	11	11	11
7	0	1	4	8	11	13	14	15	15	15
8	0	1	5	10	15	18	20	21	22	22
9	0	1	5	12	18	23	26	28	29	30

Table 1: Integer partitions of n into at most k parts, $p_{\leq k}(n)$.

Definition 5. Let $n \geq k > 0$ and define $P_k([n])$ to be the set of integer partitions of n into exactly k parts and $P_{\leq k}([n])$ to be the set of integer partitions of n into *at most* k parts. Now let $p_k(n) = |P_k([n])|$ and $p_{\leq k(n)} = |P_{\leq k}([n])|$. As usual, let $p_{\leq k}(n) = 0$ if either $n < 0$ or $k < 0$ and let $p_{\leq 0}(0) = 1$.

Example 6.

$$\begin{aligned} P([6]) &= \{(6), (5,1), (4,2), (4,1^2), (3^2), (3,2,1), (3,1^3), (2^3), (2^2,1^2), (2,1^4), (1^6)\} \\ P_2([6]) &= \{(5,1), (4,2), (3^2)\} \\ P_{\leq 3}([6]) &= \{(6), (5,1), (4,2), (4,1^2), (3^2), (3,2,1), (2^3)\} \end{aligned}$$

It is pretty easy to see that $p_2(6) = 3$, and $p_{\leq 3}(6) = 7$.

Table 1 lists a few values of $p_{\leq k}(n)$. Notice the row entries eventually stabilize.

We state a few facts about $p_{\leq k}(n)$ and $p_k(n)$ in the following proposition.

Proposition 7.

$$p_k(n) = p_{\leq k}(n) - p_{\leq k-1}(n) \tag{4}$$

and for $n > 0$,

$$p_{\leq k}(n) = p_{\leq k-1}(n) + p_{\leq k}(n-k) \tag{5}$$

Proof: The proof of identity (4) is routine. For (5), the left-hand side counts the number of integer partitions of n into at most k parts. The first term on the right-hand side counts the number of partitions of n into at most $k-1$ parts. Now let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ and define π by the rule $\pi(\lambda) = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_k - 1)$ where we agree to collapse any zero-entries.

For example, if $\lambda = (4, 3, 1) \in P_3(8)$, then $\pi(\lambda) = (3, 2, 0) = (3, 2) \in P_{\leq 3}(5)$.

Then $\pi : P_k([n]) \longrightarrow P_{\leq k}([n-k])$ is a bijection and the result follows by (4). $\square$