

Lecture 1 - Calculus of Exponential Generating Functions

We summarize the Wilf rules below.

Suppose that p is a polynomial, D is the usual derivative operator and

$$G(x) \xleftrightarrow{\text{ogf}} \{g_n\}_{n \geq 0} \quad \text{and} \quad H(x) \xleftrightarrow{\text{ogf}} \{h_n\}_{n \geq 0}$$

Then we have the following rules for ordinary generating functions.

$$\textbf{Rule 1:} \quad \frac{G(x) - g_0 - g_1x - \cdots - g_{k-1}x^{k-1}}{x^k} \xleftrightarrow{\text{ogf}} \{g_{n+k}\}_{n \geq 0}$$

$$\textbf{Rule 2:} \quad p(xD)G(x) \xleftrightarrow{\text{ogf}} \{p(n)g_n\}_{n \geq 0}$$

$$\textbf{Rule 3:} \quad G(x)H(x) \xleftrightarrow{\text{ogf}} \left\{ \sum_{k=0}^n g_k h_{n-k} \right\}_{n \geq 0}$$

$$\textbf{Rule 4:} \quad G(x)^k \xleftrightarrow{\text{ogf}} \left\{ \sum_{n_1+n_2+\cdots+n_k=n} g_{n_1} g_{n_2} \cdots g_{n_k} \right\}_{n \geq 0}$$

$$\textbf{Rule 5:} \quad \frac{G(x)}{1-x} \xleftrightarrow{\text{ogf}} \left\{ \sum_{k=0}^n g_k \right\}_{n \geq 0}$$

Now let p and D be as defined above and

$$G(x) \xleftrightarrow{\text{egf}} \{g_n\}_{n \geq 0} \quad \text{and} \quad H(x) \xleftrightarrow{\text{egf}} \{h_n\}_{n \geq 0}$$

Then we have the following rules for exponential generating functions.

$$\textbf{Rule 1':} \quad D^k G(x) \xleftrightarrow{\text{egf}} \{g_{n+k}\}_{n \geq 0}$$

$$\textbf{Rule 2':} \quad p(xD)G(x) \xleftrightarrow{\text{egf}} \{p(n)g_n\}_{n \geq 0}$$

$$\textbf{Rule 3':} \quad G(x)H(x) \xleftrightarrow{\text{egf}} \left\{ \sum_{k=0}^n \binom{n}{k} g_k h_{n-k} \right\}_{n \geq 0}$$