Date	**Section**	**Exercises**** (QC - Quick Check and CE - Class Exercises)
03/12*	-	4, 5 from here. Also, see below.
03/14*	-	3, 4, 5 from here. Also, see below.
03/17*	-	1 from here. Also, see below.
03/26*	-	(Optional) 1 from here. Also, see below.
03/28*	16.2	QC - 3; CE - 5, 6; Also, see below.
03/25*	16.2	CE - 43. Also, see below.
03/27*	16.2	CE - 31 and read Dilworth's theorem. Also, see below.
03/29*	-	See below.
03/31*	16.2	CE - 5, 32-34. Also, see below.
04/02*	-	See below.
04/04*	-	See below.
04/07*	-	See below.
04/09*	16.3	CE - 8, 13, 20. Also, see below.
04/16*	-	See below.

02/24

1. Consider the following orthogonality identity.

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{n-k} = \delta_n(m) \tag{1}$$

   (a) There is a symmetric version of (1). State it.
   (b) Use the Stirling Inversion Theorem (Theorem 2 here) to prove (1).
   (c) In Math 481 we proved (2). See Example 5 here.

$$x^n = \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} x^{\underline{k}} \tag{2}$$

   We also proved the next result. See (7) here.

$$x^{\overline{n}} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k \tag{3}$$

   Now use (2) to prove the following

$$x^n = \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} (-1)^{n-k} x^{\overline{k}} \tag{4}$$

   (d) Use the identities (3) and (4) to prove (1).
   (e) Now use (1) (or part (a)) to prove the Stirling Inversion Theorem.

2. Reprove the Binomial Inversion Theorem (Equation (2) here) as indicated below.
   (a) Let $f(x) = \sum_n f_n\, x^n/n!$ and $g(x) = \sum_n g_n\, x^n/n!$ and mimic the proof of Theorem 2 shown here.
   (b) Let $f(x) = \sum_n f_n\, x^n$ and $g(x) = \sum_n g_n\, x^n$ and once again mimic the proof of Theorem 2 shown here.

02/26

1. Show that
$$x^{\overline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^{\underline{n}} \tag{5}$$
and
$$x^{\underline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} x^{\overline{n}} \tag{6}$$

2. Prove that
$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = \sum_{j} \begin{bmatrix} n \\ j \end{bmatrix} \begin{Bmatrix} j \\ k \end{Bmatrix} \tag{7}$$

3. If $n \geq k \geq 1$, prove that
$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = \binom{n-1}{k-1} \frac{n!}{k!} \tag{8}$$

02/28

1. Find a combinatorial proof of (7) from 02/26.
Hint: $\begin{bmatrix} n \\ j \end{bmatrix}$ counts the number of ways to seat n knights at j nonempty round tables and $\begin{Bmatrix} j \\ k \end{Bmatrix}$ counts the number of ways to distribute these j tables into k nonempty rooms. Both the tables and rooms are indistinguishable.
2. Find a combinatorial proof of
$$\sum_{k} \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^k = (-1)^n \delta_n(m)$$
Hint: Using the hint given in the previous exercise, let $\mathcal{E}$ contain all seating arrangements with an even number of tables and let $\mathcal{O}$ contain all seating arrangements with an odd number of tables. Now find a bijection between $\mathcal{E}$ and $\mathcal{O}$ that has two exceptions.
3. Prove that
$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{0<j_1<j_2<\cdots<j_{n-k}<n} j_1 j_2 \cdots j_{n-k}$$
Hint: Divide both sides of (3) by x and notice that the left-hand side is the product $(x+1)(x+2)\cdots(x+n-1)$. Now compare the coefficient of x^{k-1} on the left and right-hand sides of the resulting identity.
4. Referring to Example 3 here.
   (a) Verify equations (9) and (13).
   (b) Prove that
$$\frac{k}{n}\binom{n}{k} + \frac{k+1}{n}\binom{n}{k+1} = \binom{n}{k}$$
5. Use LIF to show that
$$b_n = \sum_{k} \binom{k}{n-k} a_k \quad \text{iff} \quad a_n = \frac{1}{n} \sum_{k} \binom{2n-k-1}{n-k} k b_k (-1)^{n-k}$$
Hint: Follow Example 3 from here.

03/10

1. Let $f(x) = \sum_{n\geq 1} f_n\, x^n \in x\,\mathbb{C}[[x]],\ f_1 \neq 0$. For any $g(x) \in \mathbb{C}((x))$, define the degree of $g(x)$ as we did for formal power series. That is, $\deg(g(x)) = \min\{n \in \mathbb{Z} \,|\, [x^n]g(x) \neq 0\}$. Now let $k > 0$. Show that $f(x)^{-k} \in \mathbb{C}((x))$ with $\deg(f(x)^{-k}) = -k$.

2. Confirm the (**) step in the first proof of LIF here (page 2).

03/12

1. Suppose that $z = z(x)$ satisfies $z = x\phi(z)$. For $n \geq 0$, show that

$$[z^n]\phi(z)^n = [x^n]\left\{\frac{xz'(x)}{z(x)}\right\} = [x^n]\frac{1}{1 - x\phi'(z(x))} \tag{9}$$

Solution:

The direct proof is routine. As an alternative, we have

$$\begin{aligned}[z^n]\phi(z)^n &= [z^{n-1}]\frac{1}{z}\,\phi(z)^n \\ &= n[x^n]\int \frac{dy}{y}\,\Big|_{y=z(x)}\end{aligned}$$

where we invoked the Lagrange Inversion formula backwards. And we can proceed as we did for (13) in Problem 03 below.

2. Let $g_n = [x^n](1 + x + x^2)^n,\ n \geq 0$. Use the previous exercise to show that

$$g_n = [x^n]\frac{1}{\sqrt{1 - 2x - 3x^2}} \tag{10}$$

3. Show the following. *Hint:* For (11) use the generalized Binomial theorem.

$$\frac{1}{\sqrt{1-4x}} = \sum_{n\geq 0}\binom{2n}{n}x^n \tag{11}$$

$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^k = \sum_{n\geq 0}\frac{k(2n+k-1)!}{n!(n+k)!}\,x^n \tag{12}$$

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^k = \sum_{n\geq 0}\binom{2n+r}{n}x^n \tag{13}$$

Solution: For (11) we have

$$\frac{1}{\sqrt{1-4x}} = (1 + (-4x))^{-1/2} = \sum_{n\geq 0}\binom{-1/2}{n}(-4x)^n = \cdots$$

We leave the details to the student.

For (12), we let $C(x) = (1 - \sqrt{1-4x})/(2x)$ and let $z(x) = C(x) - 1$. Then as we have shown before (see Example 2),

$$z = x(1+z)^2 = x\phi(z) \tag{14}$$

Now let $W(z) = (1+z)^k$, then by the Lagrange Inversion formula

$$\begin{aligned}
[x^n]C(x)^k &= [x^n]W(z(x)) \\
&= \frac{1}{n}[z^{n-1}]W'(z)\phi(z)^n \\
&= \frac{k}{n}[z^{n-1}](1+z)^{k-1}(1+z)^{2n} \\
&= \frac{k}{n}[z^{n-1}](1+z)^{2n+k-1} \\
&= \frac{k}{n}\binom{2n+k-1}{n-1}
\end{aligned}$$

For (13), we once again use the Lagrange Inversion formula (step (*) below), but in the reverse direction. Let $z(x),\ C(x),$ and $\phi(z)$ be as shown above and let $g(x) = \sum_{n\geq 0}\binom{2n+r}{n}x^n$. Then

$$\begin{aligned}
[x^n]g(x) = \binom{2n+r}{n} &= [z^n](1+z)^{2n+r} \\
&= [z^{n-1}]\frac{(1+z)^r}{z}(1+z)^{2n} \\
&= [z^{n-1}]\frac{(1+z)^r}{z}\phi(z)^{2n} \\
&\overset{*}{=} n[x^n]\int \frac{(1+y)^r}{y}\,dy\,\bigg|_{y=z(x)} \\
&= [x^n]xD_x\int \frac{(1+y)^r}{y}\,dy\,\bigg|_{y=z(x)} \\
&= [x^{n-1}]\frac{(1+z)^r}{z}\,\frac{dz}{dx}\,\bigg|_{z=x\phi(z)} \qquad (15)
\end{aligned}$$

Now by (14),

$$\frac{dz}{dx} = \phi(z) + x\phi'(z)\,\frac{dz}{dx}$$

Rearranging produces

$$\frac{dz}{dx} = \frac{\phi(z)}{1-x\phi'(z)}$$

Inserting this into (15) yields

$$\begin{aligned}
\binom{2n+r}{n} &= [x^{n-1}]\frac{(1+z)^r}{z}\,\frac{\phi(z)}{1-x\phi'(z)}\,\bigg|_{z=x\phi(z)} \\
&= [x^{n-1}]\frac{\phi(z)}{z}\,\frac{(1+z)^r}{1-x\phi'(z)}\,\bigg|_{z=x\phi(z)} \\
&= [x^{n-1}]\frac{1}{x}\,\frac{(1+z)^r}{1-x\phi'(z)}\,\bigg|_{z=x\phi(z)} \\
&= [x^n]\frac{(1+z)^r}{1-x\phi'(z)}\,\bigg|_{z=x\phi(z)}
\end{aligned}$$

Now since $\phi'(z) = 2(1+z)$ and since $1 + z(x) = C(x)$, the last expression above produces

$$\begin{aligned}\binom{2n+r}{n} &= [x^n]\frac{C(x)^r}{1-2xC(x)} \\ &= [x^n]\frac{C(x)^r}{\sqrt{1-4x}}\end{aligned}$$

which is equivalent to (13).

03/14

1. Let $m_0 = 1$ and for $n > 0$, suppose that

$$m_n = m_{n-1} + \sum_{k=2}^{n} m_{k-2} m_{n-k} \tag{16}$$

Show that if $M(x) = \sum_{n\geq 0} m_n\, x^n$, then $M(x)$ satisfies the functional equation

$$M(x) - 1 = xM(x) + x^2 M(x)^2 \tag{17}$$

Solution:

For $n \geq 2$, we have

$$[x^n](M(x)-1) = m_n$$

and

$$\begin{aligned}[x^n]\left(xM(x) + x^2M(x)^2\right) &= m_{n-1} + [x^{n-2}]M(x)^2 \\ &= m_{n-1} + [x^{n-2}]\sum_{p\geq 0}\sum_{k=0}^{p} m_k\, m_{p-k} x^p \\ &= m_{n-1} + \sum_{k=0}^{n-2} m_k\, m_{n-2-k} \\ &= m_{n-1} + \sum_{k=2}^{n} m_{k-2}\, m_{n-k}\end{aligned}$$

So by (16),

$$[x^n](M(x)-1) = [x^n]\left(xM(x) + x^2M(x)^2\right)$$

for $n \geq 2$. The cases when $n \in \{0, 1\}$ are trivial and are left as exercises. The result now follows.

2. Find the sum of the first n terms in the binomial expansion of

$$\left(1-\frac{1}{2}\right)^{-n} = \sum_{k\geq 0}\binom{-n}{k}\left(\frac{-1}{2}\right)^k = \sum_{k\geq 0}\binom{n+k-1}{k}2^{-k} \tag{18}$$

For example, when $n = 4$ the sum is $1 + 4/2 + 10/4 + 20/8 = 8$. *Hint:* Use LIF.

3. Let $J(x) = (1+x)^2/(2+x)$. Show that for all $n \in \mathbb{P}$

$$[x^{n-1}]\frac{J(x)^n}{1+x} = \frac{1}{2}$$

03/17

1. Let $\{a_n\}_{n\geq 0} \subset \mathbb{R}$ with $a_0 \neq 0$. Find a sum formula for $[z^n]\left(\sum_{k=0}^{N} a_k\, z^k\right)^n$ when $N \in \{2,3\}$. Do you see a pattern?

2. Let $\mathcal{T} = \mathcal{T}^{\Omega}$ where $\Omega = \{0,1,3\}$. However, this time we measure the size of each tree by the number of edges. Let $T(x)$ be the ordinary generating function for $\mathcal{T}$. Find a sum formula for $[x^n]T(x)$.

3. Let m_n be the Motzkin numbers as defined on page 3 here and let $\{c_n\}_{n\geq 0}$ be the Catalan numbers. Answer the questions below.
   (a) Use (17) to show that
   $$M(x) = \sum_{n\geq 0} m_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}$$
   (b) Show that
   $$m_n = \sum_k \binom{n}{2k} c_k \quad \text{and} \quad c_{n+1} = \sum_k \binom{n}{k} m_k$$
   Hint: Part (a) above and exercise 3 from 02/17 should help with the second identity.
   (c) Show the Motzkin's original definition (stated here) is equivalent to the one given in class by showing that the original definition satisfies the following recursion.
   $$m_n = m_{n-1} + \sum_{k=2}^{n} m_{k-2} m_{n-k}, \quad n > 0$$

4. Find a formula t_n for the number of triangulations of an $(n+2)$-gon. (e.g., $t_1 = 1$ and $t_2 = 2$ since there is one triangulation of a triangle and there are two triangulations of a square).

03/19

1. Consider the *lattice of compositions*, $(K_n, \leq)$. Here K_n is the set of all compositions of n and $\alpha \leq \beta$ is a refinement of compositions defined by
   If $[\alpha_1, \alpha_2, \ldots, \alpha_p] \vDash \alpha$ and $[\beta_1, \beta_2, \ldots, \beta_q] \vDash \beta$, then $[\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_l}] \vDash \beta_k$ for $k \in [q]$.
   For example, in K_{11}, $3+2+5+1$ is a refinement of $5+5+1$ hence $[3,2,5,1] \leq [5,5,1]$. On the other hand, $[3,3,4,1] \not\geq [5,5,1]$. Sketch the Hasse diagram for K_4.

2. The *Young lattice* $(Y, \leq)$ is the set of all integer partitions and $\alpha \leq \beta$ if the Young diagram for α is a contained in the Young diagram for β. Sketch the Hasse diagram for Y up to integer partitions of 4.

03/21

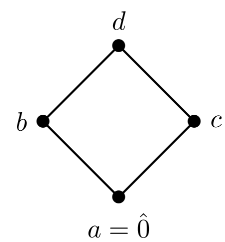
1. Find all linear extensions (see Example 16.9 of the text) of the 5 posets shown in Figure 16.3 from the text.

2. List all 4-element posets.

3. How many linear extensions do the posets below have?

03/24

1. Consider the poset P shown below and the linear extension $L(a) = 1, L(b) = 3, L(c) = 2, L(d) = 4$ to answer the questions that follow.



(a) Let $Z = Z_\zeta$ be the upper-triangular matrix associated with zeta function ζ_P of P. Find Z.

(b) Use a CAS to find the matrix $M = M_\mu$ associated with the Möbius function μ_P of P.

(c) Now let $\mu(x) = \mu(a, x)$ and compute $\mu(x)$ for all $x \in P$. Compare to the values that we obtained in class using the linear extension $K(a) = 1, K(b) = 2, K(c) = 3, K(d) = 4$.

2. Repeat the previous exercise for the divisor lattice D_{30}. If you are working with a classmate, choose different linear extensions and compare results.

3. Let $P = \{a, b, c, d, e, f\}$ be a poset with linear extension $L : P \to [6]$ defined by $L(a) = 1, L(b) = 2, \ldots, L(f) = 6$. Suppose that Z_L, the zeta matrix of P, is defined as

$$Z_L = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Answer the questions below.

(a) Use Z_L to sketch the Hasse diagram for P.

(b) Use the diagram that you created in part (a) to compute $\mu(a, y)$ for all $y \in P$. Also, explain how to use this link to check your values.

03/26

1. For each of the following posets $(P, \leq)$, sketch the Hasse diagram and use Theorem 16.15 from the text to compute $\mu(x) := \mu(\hat{0}, x)$ for all $x \in P$.
   (a) $P = 2^{[4]}$ and the partial order is set containment. That is, $x \leq y$ if $x \subseteq y$.

   (b) $P = \Pi_4$, the (set) partition poset. Here the partial order is "refinement". That is, $x \leq y$ if each block in x is contained in a block in y. For example, $1/2/34 \leq 12/34$.

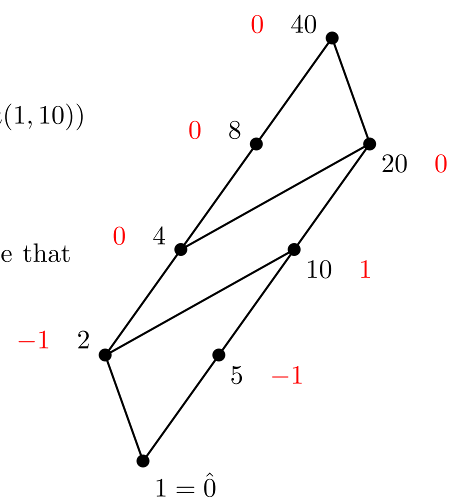
   (c) $P = D_{40}$, the divisor lattice with the usual partial order.

   Solution:

   The Mobius function values are shown in red. For example,

$$\begin{aligned}\mu(20) = \mu(1,20) &= -(\mu(1,1) + \mu(1,2) + \mu(1,5) + \mu(1,4) + \mu(1,10)) \\ &= -(1 - 1 - 1 + 0 + 1) \\ &= 0\end{aligned}$$

   Notice that we can also appeal directly to 16.20 to conclude that $\mu(1, 20) = 0$ since $\frac{20}{1}$ is not square-free.



2. Construct the ζ matrix Z for the divisor lattice D_{40} and use a CAS to find the μ matrix M. Compare the first row of M with the values derived from the exercise above.

   Solution:

   Note: The first row of the ζ matrix Z just gives the linear order that was used to compute the entries (of Z) and is not actually part of the matrix.

$$Z = \begin{pmatrix} 1 & 2 & 4 & 5 & 8 & 10 & 20 & 40 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   Compare the first row of M with the Mobius function values displayed in exercise 1(c) above.

03/28

1. Read the proof of Theorem 7.6 in the text.

2. Verify the statement that the intervals $[x, y]$ and $[1, y/x]$ are isomorphic as posets in Example 16.20 of the text.

03/31

1. Use the Theorem 7.6 to re-prove Theorem 2 in the Binomial Inversion handout.

   Solution:

2. Let P be the poset of the positive integers with $x \le y \in P$ if $x \mid y$. Also, let $p_1, p_2, \ldots, p_k$ be k distinct primes and let $y = p_1 \cdot p_2 \cdots p_k$. Show that $[1, y]$ is isomorphic to B_k.

   Solution:

   Let $f : B_k \to [1, y]$ be defined as follows. $f(\emptyset) = 1$ and for $\emptyset \ne T \subseteq [k]$, let $f(T) = \prod_{m \in T} p_m$. It is easy to confirm that f is a bijection. Also, f is order-preserving, for if $T \subseteq S \subseteq [k]$, then

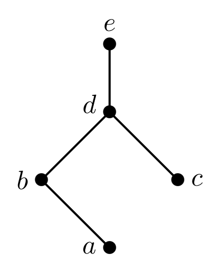
   $$f(T) = \prod_{m \in T} p_m \quad \text{and} \quad f(S) = \prod_{m \in S} p_m$$

   Now it is easy to conclude that $f(T) \mid f(S)$. That is, $f(T) \le f(S)$, as expected.

   Note: The conclusion is false if the primes $p_1, p_2, \ldots, p_k$ are not distinct. Where did we use this fact in the proof?

3. Let P be the poset shown below and consider the linear extension $(x_1, x_2, x_3, x_4, x_5) = (c, a, b, d, e)$. Let $f : P \to \mathbb{R}$ be defined be $f(c) = f(x_1) = -3, f(a) = 1, f(b) = 2, f(d) = 6, f(e) = 10$. Finally, let $g : P \to \mathbb{R}$ be given by

   $$g(y) = \sum_{x \le y} f(x) \tag{19}$$

   (a) Construct the zeta matrix Z associated with this linear extension. *Note:* It should be different than the matrix that we discovered in class. Also, use a CAS to find the Mobius matrix M.

   (b) Use the matrix Z to determine the values of the function g defined in (19). *Compare with the values that we computed in class.*

   (c) Let $\boldsymbol{f} = [f(x_1)\ f(x_2)\ f(x_3)\ f(x_4)\ f(x_5)]$ and similarly for $\boldsymbol{g}$. Confirm that $\boldsymbol{f} = \boldsymbol{g}M$.

04/02

1. Let P and Q be posets. Show that $P \times Q$ with partial order as given by Definition 16.23 is a poset.

2. Construct a poset P such that $\mu(\hat{0}, x) = n$ for any $n \in \mathbb{Z}$.

3. Let P and Q be posets and consider the following alternative (partial) orders on $P \times Q$. Is $P \times Q$ a poset under the given order? *Note:* Throughout, we assume that $[p, p'] \subset P$ and $[q, q'] \subset Q$ and, for example, we write $p \le p'$ instead of $p \le_P p'$, etc.

   (a) $(p, q) \le (p', q')$ if $p < p'$ or if $p = p'$ and $q \le q'$.

   (b) $(p, q) \le (p', q')$ if $p \le p'$.

(c) $(p,q) \le (p',q')$ if $p<p'$ and $q<q'$ or $p=p'$ and $q=q'$.

04/04 The exercises below depend on the following results.

Proposition. Let $[x,y]$ be an interval in Π_n with the usual refinement (partial) order. If $y = B_1/B_2/\cdots/B_k$ and if each B_i splits into n_i blocks in x, then

$$[x,y] \cong \prod_{i=1}^{k} \Pi_{n_i} \tag{20}$$

In particular,

$$\mu(x,y) = \prod_{i=1}^{k} \mu(\Pi_{n_i}),$$

by Theorem 16.24. For example, let $x = 1/3/256/47$ and $y = 1347/256$ in Π_7. Then $x<y$ and

$$\begin{aligned}\mu(x,y) &= \mu(\Pi_3)\mu(\Pi_1)\\ &= (-1)^2 2! \cdot (-1)^0 0! = 2\end{aligned}$$

And the last line follows since

$$\mu(\Pi_n) = (-1)^{n-1}(n-1)! \tag{21}$$

as we showed in class.

1. Use the above results to compute $\mu(x,\hat{1})$ for all $x \in \left\{ {[4] \atop k} \right\}$ for each $k \in [3]$. Also, compute $\mu(13/2/48/56/7, 123478/56)$ and $\mu(13/2/48/56/7, \hat{1})$ in Π_8.

2. Let $\{f_n\}_{n\ge1}$ where $f_n = 2C_n - n$ and C_n are the Catalan numbers. Let Π_n be the set partition poset with the usual refinement order. Let $F : \Pi_4 \to \mathbb{Z}$ be defined by the rule $F(x) = f_{5-|x|}$ where $|x|$ is equal to the number of blocks in x. If we define $G(y) = \sum_{x\le y} F(x)$, then by Mobius inversion

$$F(y) = \sum_{x\le y} G(x)\mu(x,y) \tag{22}$$

Use (22) to show that $F(1234) = 24$.

04/07

1. Find an interval $[x,y] \subset \Pi_n$ such that

   (a) $\mu(x,y) = -12$

   (b) $\mu(x,y) = 96$

   Note: In each case, you will also need to specify the value of n. Answers will not be unique.

2. In our textbook's the definition of the *incidence algebra*, $I(P)$, it is stated that P must be a locally finite poset. Why is this?

3. Prove (20) in the proposition stated at the beginning of the assignments from 04/04.

4. Show that $\prod_n$ is a lattice. In addition, verify the following

   Suppose that $\rho, \tau \in \prod_n$. Then $\delta = \rho \vee \tau$ is the partition such that b and c are in the same block of δ if and only if these is a sequence of blocks $B_1, B_2, \ldots, B_m$ where each B_i is a block of either ρ or τ, $b \in B_1$, $c \in B_m$, and $B_i \cap B_{i+1} \neq \emptyset$ for all i.

04/09

1. Let L be a lattice with $x, y, z \in L$. Prove the following statements.
   (a) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$

   (b) $x \leq y \Longleftrightarrow x \wedge y = x \Longleftrightarrow x \vee y = y$

   (c) $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$

2. If L is a lattice and M is a poset and if there is an isomorphism $f : L \to M$, then M is also a lattice. In addition, for $x, y \in L$,

$$f(x \wedge y) = f(x) \wedge f(y)$$

   Note: Recall that poset isomorphisms are, by definition, order-preserving.

04/16

1. Let $A_0 = \sum_{n\geq 1} a_{2n}x^n$ and let $A_1 = \sum_{n\geq 0} a_{2n+1}x^n$. In class we showed that $a_{2n+1} = \sum_{j=0}^{n} a_{2j}a_{2n-2j}$. Show that

$$A_1 = (1 + A_0)^2$$

2. Let $a_n = 2 \cdot 3^n$ for all $n \in \mathbb{Z}$. Recall that $A(x) = \sum_{n\geq 0} a_n x^n = 2(1 - 3x)^{-1}$. Let $B(x) = \sum_{n\geq 1} a_{-n}x^n$. Is there any relationship between $A(x)$ and $B(x)$? More specifically, is there a transformation T that such that $B(x) = T(A(x))$?

3. How many acyclic orientations are there for each of the graphs below? In each case, sketch a few.
   (a) K_3

   (b) P_3

   (c) C_4