Date	**Section**	**Exercises**** (QC - Quick Check and CE - Class Exercises)
01/13*	8.2	CE - 31, 32
01/15*	8.2	CE - 24, 45, 46
01/17*	5.3	CE - 7, 8, 11, 14
01/22*	5.3	CE - 30
01/24*	-	See below.
01/27*	-	See below.
01/29*	-	See below.
01/31*	-	See below.
02/03	-	2, 4, 7 from here.
02/05*	-	See below.
02/07*	-	See below.
02/10*	-	See below.
02/12*	-	See below.
02/17*	-	1(c), 2, 4, from here. Also, see below.

01/13

1. Extend Rule 3′ to a product of 3 exponential generating functions. Verify your formula.

   Solution:

   Let $A \xleftrightarrow{\text{egf}} \{a_n\}_n$, $B \xleftrightarrow{\text{egf}} \{b_n\}_n$, and $C \xleftrightarrow{\text{egf}} \{C_n\}_n$. Then

$$
\begin{aligned}
A(x)B(x)C(x) &= \sum_{n\geq 0} \underbrace{\sum_k \binom{n}{k} a_k b_{n-k}}_{h_n} \frac{x^n}{n!}\, C(x) = \sum_{n\geq 0} h_n \frac{x^n}{n!}\, C(x) \\
&= \sum_{n\geq 0} \sum_j \binom{n}{j} h_j c_{n-j} \frac{x^n}{n!} \\
&= \sum_{n\geq 0} \sum_j \binom{n}{j} \sum_k \binom{j}{k} a_k b_{j-k}\, c_{n-j} \frac{x^n}{n!} \\
&= \sum_{n\geq 0} \sum_{j=0}^{n} \sum_{k=0}^{j} \frac{n!}{k!(j-k)!(n-j)!}\, a_k b_{j-k}\, c_{n-j} \frac{x^n}{n!} \\
&= \sum_{n\geq 0} \sum_{i+j+k=n} \frac{n!}{i!j!k!}\, a_i b_j c_k \frac{x^n}{n!}
\end{aligned}
$$

2. Recall that $\pi \in \mathfrak{S}_n$ is called an <u>involution</u> if $\pi^2 = \text{id}$. Let i_n count the number of involutions in $\mathfrak{S}_n$ (the set of all permutations on $[n]$) and let $i_0 = 1$.

   (a) Show that $i_1 = 1$ and for $n \geq 0$,
$$i_{n+2} = i_{n+1} + (n+1)i_n \tag{1}$$

   Solution:

**Exercises from the *A Walk Through Combinatorics, 4*th *ed.*, Miklós Bóna, World Scientific

An involution must consist entirely of 1-cycles and 2-cycles. Now the left-hand side counts the number of involutions on $[n+2]$. For the right-hand side, there are i_{n+1} involutions with $n+2$ in a 1-cycle. Otherwise, there are $\binom{n+1}{1} = n+1$ ways to choose the element paired with $n+2$ and i_n ways to permute the remaining items (as an involution). So by the product rule, there are $(n+1)i_n$ ways that $n+2$ can be in a 2-cycle. Since these cases are mutually exclusive, the result now follows by the sum rule.

(b) Show that

$$\sum_{n\geq 0} i_n \frac{x^n}{n!} = e^{x+x^2/2}$$

Solution:

Let $A(x) = \sum_n i_n\, x^n/n!$. According to the Wilf rules, the recursion (1) is equivalent to the following differential equation

$$\begin{aligned} A''(x) &= A'(x) + (xD + I)A(x) \quad (D = \text{derivative operator and } I = \text{identity map}) \\ &= (x+1)A'(x) + A(x) = D((x+1)A(x)) \end{aligned}$$

Integrating both sides yields

$$A'(x) = (x+1)A(x) + C \qquad (\text{but } C = 0 \text{ since } A'(0) = A(0) = 1)$$

Rearranging and integrating gives

$$\begin{aligned} \frac{A'(x)}{A(x)} &= 1 + x \\ \ln A(x) &= x + x^2/2 + C \qquad (\text{and once again } C = 0 \text{ since } A(0) = 1) \end{aligned}$$

The result now follows.

3. Let $\{f_n\}_{n\geq 0}$ be a sequence and let Δ be the forward difference operator. That is, $\Delta f_j = f_{j+1} - f_j$. Show that

$$\Delta^n f_0 = \sum_k \binom{n}{k} f_k (-1)^{n-k} \tag{2}$$

Note: $\Delta^n f_k = \Delta\,\Delta^{n-1} f_k$ and $\Delta^0 f_k = f_k$.

Solution:

**Exercises from the *A Walk Through Combinatorics*, 4^{th} *ed.*, Miklós Bóna, World Scientific

We induct on n. Clearly (2) holds when $n = 0$ since both sides produce f_0. Now suppose that (2) holds. Then

$$\begin{aligned}
\Delta^{n+1} f_0 = \Delta\,\Delta^n f_0 &= \Delta \sum_{k=0}^{n} \binom{n}{k} f_k (-1)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} (f_{k+1} - f_k)(-1)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} f_{k+1}(-1)^{n-k} - \sum_{k=0}^{n} \binom{n}{k} f_k (-1)^{n-k} \\
&= f_{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} f_k (-1)^{n+1-k} + \sum_{k=1}^{n} \binom{n}{k} f_k (-1)^{n+1-k} + (-1)^{n+1} f_0 \\
&= f_{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k} \right) f_k (-1)^{n+1-k} + (-1)^{n+1} f_0 \\
&= f_{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} f_k (-1)^{n+1-k} + (-1)^{n+1} f_0 \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f_k (-1)^{n+1-k}
\end{aligned}$$

as expected.

01/15

1. Let $c_0 = 1$ and for $n > 0$ let c_n count the number of n-permutations in which each cycle is colored red, green, or blue.

   (a) Find a sum formula for c_n.

   Solution:

   Let $\pi \in \begin{bmatrix} [n] \\ k \end{bmatrix}$. Then π can be colored in 3^k ways. So by the product rule, there are $\begin{bmatrix} n \\ k \end{bmatrix} 3^k$ ways to color n-permutations that consist of exactly k cycles. Summing across k yields

   $$c_n = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} 3^k$$

   (b) Find a simple factorial formula for c_n.

   Solution:

   Manual computation using the above formula produces the sequence $1, 3, 12, 60, \ldots$. So we guess $c_n = (n+2)!/2,\ n \geq 0$. Fortunately, we don't have to guess. In Math 481 we showed that

   $$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k = x^{\overline{n}} = x(x+1)\cdots(x+n-1)$$

   After the substitution $x = 3$, we obtain

   $$\begin{aligned}
   c_n &= 3(3+1)\cdots(3+n-1) \\
   &= \frac{2}{2}\,\frac{3(3+1)\cdots(2+n)}{1} = \frac{(n+2)!}{2}
   \end{aligned}$$

**Exercises from the *A Walk Through Combinatorics, 4th ed.*, Miklós Bóna, World Scientific

(c) Let $C(x) = \sum_n c_n x^n/n!$. Find the closed form of $C(x)$.

Solution:

$$\begin{aligned} C(x) &= \sum_{n\geq 0} \frac{(n+2)!}{2}\,\frac{x^n}{n!} \\ &= \frac{1}{2}\sum_{n\geq 0}(n+2)(n+1)x^n \\ &= D^2\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^3} \end{aligned}$$

(d) Now let $a_0 = a_1 = 1$ and let $a_{n+2} = c_n$ for $n \geq 0$. Find the closed form for $A(x) = \sum_n a_n x^n/n!$. *Note:* I will explain the reason for this part later.

Solution:

According to the Wilf rules, $A''(x) = C(x)$. It follows that $A(x) = (1-x)^{-1}$.

2. A coach wishes to break up her n-member team into 3 practice squads. Players on squad A will wear either red, white, or blue jerseys, those on squad B will wear yellow or green jerseys, and squad C players will wear black jerseys. Let $t_0 = 1$ and for $n > 0$, let t_n count the number of ways that she can do this.

(a) Find a simple formula for t_n.

Solution:

There are 6 jersey colors, so this should just be 6^n.

(b) Let $T(x) \xleftrightarrow{\text{egf}} \{t_n\}$. Find the closed form of $T(x)$ and use it to confirm your answer in part (a).

Solution:

Let $i, j,$ and k be the number of players resp. on squad A, squad B, and squad C. Then

$$t_n = \sum_{i+j+k=n} \frac{n!}{i!j!k!}\, 3^i\, 2^j\, 1^k$$

So by the Wilf rules, we must have

$$\begin{aligned} T(x) = \sum_n t_n \frac{x^n}{n!} &= \sum_n 3^n \frac{x^n}{n!} \sum_n 2^n \frac{x^n}{n!} \sum_n \frac{x^n}{n!} \\ &= e^{3x}\, e^{2x}\, e^x = e^{6x} \end{aligned}$$

as expected.

(c) In addition to the initial conditions, suppose also that squad B has a captain and players on squad C wear numbered black jerseys. Find the closed form for $T(x)$ in this case.

Solution:

$$t_n = \sum_{i+j+k=n} \frac{n!}{i!j!k!} 3^i \, j2^j \, k!$$

So by the Wilf rules, we must have

$$\begin{aligned} T(x) = \sum_n t_n \frac{x^n}{n!} &= \sum_n 3^n \frac{x^n}{n!} \sum_n n2^n \frac{x^n}{n!} \sum_n n! \frac{x^n}{n!} \\ &= e^{3x} \, 2xe^{2x} \, \frac{1}{1-x} = \frac{2xe^{5x}}{1-x} \end{aligned}$$

The first few terms of this sequence are

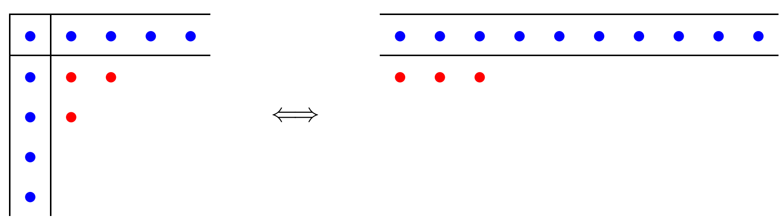
$$0, 2, 24, 222, 1888, 15690, 131640, 1140230, 10371840, \ldots$$

01/17 Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ and define $\pi : P_k([n]) \to P_{\leq k}([n-k])$ by $\pi(\lambda) = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_k - 1)$. Here we agree to collapse any zero entries. Show that π is a bijection.

01/22

1. We say that an integer partition λ is self-conjugate if $\lambda = \lambda^t$. Show that the number of self-conjugate $\lambda \vdash n$ is equal the number of $\mu \vdash n$ having distinct parts and odd. *Hint:* Use Young diagrams to find a bijection between the collection of self-conjugate partitions $P_{\text{elf}}([n])$ and the collection of partitions with distinct parts and odd, call it $P_{\text{do}}([n])$.

   Solution:

2. For $n \geq m \geq 0$, show that

$$\sum_{k=0}^{m} \binom{m}{k} k^n (-1)^k = (-1)^m m! \left\{ {n \atop m} \right\} \tag{3}$$

   Solution:

Recall that the Stirling numbers of the second kind can be defined as the numbers that allow powers of x to expressed in terms of the falling factorial. We have

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}} \tag{4}$$

Thus

$$\begin{aligned}
\sum_k \binom{m}{k} k^n (-1)^k &= \sum_{k=0}^{m} \binom{m}{k} \sum_j \left\{ {n \atop j} \right\} (-1)^k k^{\underline{j}} \qquad \text{(by (4))} \\
&= \sum_j j! \left\{ {n \atop j} \right\} \sum_k \binom{m}{k} \binom{k}{j} (-1)^k \\
&= \sum_j j! \left\{ {n \atop j} \right\} \delta_{jm} (-1)^m \\
&= (-1)^m m! \left\{ {n \atop m} \right\}
\end{aligned}$$

3. Let $D(x) = \prod_{j \geq 0} (1 + x^{2^j})$. Find a combinatorial proof that $D(x) = (1-x)^{-1}$.
   Hint: Show that $[x^n]D(x) = 1$ for all n.

   Solution:

   The basic idea is that there is exactly one way to write any positive integer in base 2.

4. Let $g(n)$ count the number of partitions of n that have no part equal to 1 or 2. Express $g(n)$ in terms of $p(n)$.

   Solution:

   Observe that this implies that $n \geq 3$. Now let $G(x) = \sum_n g(n) x^n$. Then

$$\begin{aligned}
G(x) = \sum_{n \geq 0} f(n) x^n &= \frac{1}{1-x^3} \cdot \frac{1}{1-x^4} \cdots \\
&= \frac{1-x}{1-x} \cdot \frac{1-x^2}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^4} \cdots \\
&= (1 - x - x^2 + x^3) \mathcal{E}(x)
\end{aligned}$$

   It follows that

$$\begin{aligned}
g(n) &= [x^n](1 - x - x^2 + x^3)\mathcal{E}(x) \\
&= [x^n]\mathcal{E}(x) - [x^{n-1}]\mathcal{E}(x) - [x^{n-2}]\mathcal{E}(x) + [x^{n-3}]\mathcal{E}(x) \\
&= p(n) - p(n-1) - p(n-2) + p(n-3), \quad n \geq 3
\end{aligned}$$

01/24

1. **Binary Words** - Let $\mathcal{B} = \{a, b\}$ where $|a| = |b| = 1$. Find the first 6 terms in the counting sequence A_n of $\mathcal{A} = \mathrm{SEQ}(\mathcal{B})$.

2. Let $\mathcal{I} = \text{SEQ}\left(\mathcal{Z}_\bullet\right) \setminus \{\Box\}$. Find the generating function for $\mathcal{A} = \text{SEQ}(\mathcal{I})$.

 Solution:

 Notice that $\mathcal{I}$ has one object of size 1, one object of size 2, etc. It follows that its generating function is $\frac{1}{1-x} - 1 = \frac{x}{1-x}$ and the ordinary generating function of the class $\mathcal{A}$ is

$$A(x) = \frac{1}{1 - \frac{x}{1-x}} = \frac{1-x}{1-2x} \tag{5}$$

 and so, its counting sequence must be

$$\begin{aligned} [x^n]A(x) &= [x^n]\,\frac{1}{1-2x} - [x^n]\,\frac{x}{1-2x} && (6) \\ &= 2^n - 2^{n-1} = 2^{n-1} && (7) \end{aligned}$$

 In other words, $\mathcal{A}$ is combinatorially equivalent to the class of compositions since they have the same counting sequences.

3. Let $\mathcal{Z}_\bullet = \{\bullet\}$ and $\mathcal{B}_{(j,k)} = \underbrace{\mathcal{Z}_\bullet \times \cdots \times \mathcal{Z}_\bullet}_{j\text{ factors}} + \underbrace{\mathcal{Z}_\bullet \times \cdots \times \mathcal{Z}_\bullet}_{k\text{ factors}} = \mathcal{Z}_\bullet^j + \mathcal{Z}_\bullet^k$.

 (a) Find the generating function of $\mathcal{B}_{(2,5)}$ and $\mathcal{C} = \text{SEQ}\left(\mathcal{B}_{(2,5)}\right)$.

 Solution:

 We have $B(x) = x^2 + x^5$ so that

$$C(x) = \sum_n c_n\, x^n = \frac{1}{1 - x^2 - x^5}$$

 It is easy to confirm (by writing out the elements in $\mathcal{C}$) that the first few terms of coefficient sequence $\{c_n\}$ must be $1, 0, 1, 0, 1, 1, \ldots$ in agreement with this.

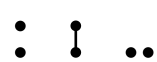
 (b) Find the generating function of $\mathcal{B}_{(1,k)}$ and $\mathcal{C} = \text{SEQ}\left(\mathcal{B}_{(1,k)}\right)$.

 Solution:

 We have $B(x) = x + x^k$ so that

$$C(x) = \sum_n c_n\, x^n = \frac{1}{1 - x - x^k}$$

 (c) In class, we showed that the generating function of $\mathcal{A} = \text{SEQ}\left(\mathcal{B}_{(1,2)}\right)$ was $A(x) = (1 - x - x^2)^{-1}$. Find the generating function for $\mathcal{C} = \text{SEQ}(\mathcal{A} \setminus \mathcal{E})$. The first few terms in the sequence of coefficients c_n are $1, 1, 3, 8, 22, 60, \ldots$. *Note:* You will need to figure out what the generating function $A_\epsilon(x)$ for the class $\mathcal{A} \setminus \mathcal{E}$ must be, but that shouldn't be too difficult since $A_\epsilon(x) = A(x) - A(0)$.
 Using the symbols $\bullet$, $\bullet\!\!-\!\!\bullet$, also list the 8 elements of size three. For example,



 are the 3 elements of size two.

 Solution:

Since $A_\epsilon(x) = \frac{1}{1-x-x^2} - 1$, the ordinary generating function for $\mathcal{C}$ is

$$C(x) = \frac{1}{1 - A_\epsilon(x)} = \frac{1-x-x^2}{1-2x-2x^2}$$

01/27

1. More on exponential generating functions.

   (a) On Quiz 1 we used the identity in (3) to find a closed form for the exponential generating function below.

$$S_k(x) = \sum_{n\geq 0} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \frac{(e^x-1)^k}{k!} \tag{8}$$

Reprove (8) using the recursion for $\left\{ {n \atop k} \right\}$. *Hint:* Try induction on k.

Solution:

Following the hint, we proceed by induction on k. For $k = 0$, we have

$$\begin{aligned} S_0(x) &= \sum_{n\geq 0} \left\{ {n \atop 0} \right\} \frac{x^n}{n!} \\ &= \left\{ {0 \atop 0} \right\} \frac{x^0}{0!} + \left\{ {1 \atop 0} \right\} \frac{x^1}{1!} + \cdots + \left\{ {n \atop 0} \right\} \frac{x^n}{n!} + \cdots \\ &= 1 + 0 + 0 + \cdots \end{aligned}$$

in agreement with (8) and the base case is established. Now suppose (8) holds for all $j < k$. Then

$$\begin{aligned} S_k'(x) &= \sum_{n\geq 0} \left\{ {n+1 \atop k} \right\} \frac{x^n}{n!} && \text{(Wilf Rule 1}'\text{)} \\ &= k \sum_{n\geq 0} \left\{ {n \atop k} \right\} \frac{x^n}{n!} + \sum_{n\geq 0} \left\{ {n \atop k-1} \right\} \frac{x^n}{n!} && \text{(by recursion)} \\ &= kS_k(x) + \frac{(e^x-1)^{k-1}}{(k-1)!} && \text{(by induction)} \end{aligned}$$

Rearranging produces the differential equation

$$S_k'(x) - kS_k(x) = \frac{(e^x-1)^{k-1}}{(k-1)!}$$

which can be evaluated by elementary techniques. We try multiplying by the integrating factor e^{-kx} to obtain

$$D_x \left(e^{-kx} S_k(x) \right) = \frac{(1-e^{-x})^{k-1}}{e^x (k-1)!}$$

Integrating both sides produces

$$e^{-kx} S_k(x) = \frac{(1-e^{-x})^k}{k!} + C \qquad \text{(but } C = 0 \text{ since } S_k(0) = 0\text{)}$$

Now this last equation is equivalent to (8).

Remark. Notice that $S_1(x) = \sum_{n\geq 1} \left\{ {n \atop 1} \right\} \frac{x^n}{n!} = \sum_{n\geq 1} \frac{x^n}{n!} = e^x - 1$. One could then determine that $S_2(x) = (e^x-1)^2/2$, as we do in part (b) below, to "guess" the general formula in (8).

(b) Find a formula for $C_k(x) = \sum_{n\geq 0} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!}$ and then use the recursion for $\begin{bmatrix} n \\ k \end{bmatrix}$ to verify your formula.

Solution:

We claim that

$$C_k(x) = \sum_{n\geq k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!} = \frac{1}{k!}\left(\ln \frac{1}{1-x}\right)^k \tag{9}$$

We outline the proof below.

(i) First recall that $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$. Thus

$$C_1(x) = \sum_n \begin{bmatrix} n \\ 1 \end{bmatrix} \frac{x^n}{n!} = \sum_{n\geq 1} \frac{x^n}{n}$$

It follows that

$$C_1'(x) = \sum_{n\geq 1} x^{n-1} = \frac{1}{1-x}$$

so that

$$C_1(x) = \ln \frac{1}{1-x}$$

(ii) Before we try to guess a general pattern, let's try to find the closed form of $C_2(x)$. Taking derivatives in part (i) turned out to be useful. If we apply Wilf Rule $1'$ together with the recursion formula for $\begin{bmatrix} n \\ k \end{bmatrix}$, we obtain

$$\begin{aligned} C_2'(x) &= \sum_n \begin{bmatrix} n+1 \\ 2 \end{bmatrix} \frac{x^n}{n!} \\ &= \sum_n n \begin{bmatrix} n \\ 2 \end{bmatrix} \frac{x^n}{n!} + \sum_n \begin{bmatrix} n \\ 1 \end{bmatrix} \frac{x^n}{n!} \\ &= xC_2'(x) + C_1(x) \end{aligned}$$

Rearranging yields the differential equation,

$$C_2'(x) = \frac{1}{1-x} \ln \frac{1}{1-x}$$

which admits the solution,

$$C_2(x) = \frac{1}{2}\left(\ln \frac{1}{1-x}\right)^2$$

(iii) We claim that the general form appears to be

$$C_k(x) = \frac{1}{k!}\left(\ln \frac{1}{1-x}\right)^k$$

The proof of this fact is nearly identical to part (ii) and we leave it as an exercise. See also part (a) above.

2. Consider the sequence $\{a_n\}$ satisfies the following recursion. $a_0 = a_1 = 1,\ a_2 = 2$ and for $n > 2$

$$a_{n+1} = (n+1)a_n - \binom{n}{2} a_{n-2}$$

The first few terms of this sequence are $1, 1, 2, 5, 17, 73, \ldots$. Show the exponential generating function $A(x) = \sum_n a_n\, x^n/n!$ satisfies the ordinary differential equation

$$(1-x)A'(x) = \left(1 - \frac{x^2}{2}\right) A(x)$$

and is given by

$$A(x) = \frac{e^{x/2+x^2/4}}{\sqrt{1-x}}$$

01/29

(a) Prove that Subset $\cong$ SEQ($\{0,1\}$) with $|0| = |1| = 1$ (see Example 1 here).

(b) Use equation (1) from here to prove the Binomial theorem. That is, prove that $(1+y)^n = \sum_k \binom{n}{k} y^k$.

(c) Convince yourself that Definition 2 from here makes sense by generating all of the terms in the expansion of the right-hand side of equation (1) for $0 \le n \le 3$.

01/31

1. List at least 8 elements in each of the following classes. Also, find the corresponding generating functions.

(a) $b\,\mathrm{SEQ}(a)$

Solution:

The ordinary generating function is

$$\frac{x}{1-x}$$

(b) $\mathrm{SEQ}(b\,\mathrm{SEQ}(a))$

Solution:

The ordinary generating function is

$$\frac{1}{1-\frac{x}{1-x}}$$

Notice that we used the generating function from part (a).

(c) $\mathrm{SEQ}(a)\,\mathrm{SEQ}(b\,\mathrm{SEQ}(a))$

Solution:

The ordinary generating function is

$$\frac{1}{1-x}\,\frac{1}{1-\frac{x}{1-x}} = \frac{1}{1-2x}$$

Notice that we used the generating function from part (b).

2. Let $\mathcal{W}^2 = \mathrm{SEQ}(a)\,\mathrm{SEQ}(b\,\mathrm{SEQ}(a))$.

   (a) Identify $\mathcal{W}^2$. List enough elements to see what is going on and find a more direct description.

   Solution:

   Should be words of arbitrary length using the alphabet $\{a, b\}$, in agreement with the generating function that we found in problem 1(c) above.

   (b) What is $\mathcal{W}^1$? Express $\mathcal{W}^3$ in two different ways.

   Solution:

   $\mathcal{W}^1 = \varepsilon + a + aa + aaa + \cdots$. In other words (no pun intended), words of arbitrary length using the alphabet $\{a\}$.

   $\mathcal{W}^3$ should be words of arbitrary length using the alphabet $\{a, b, c\}$.

01/31

1. Let $\mathcal{B} = \{\bullet, \bullet\ \bullet{-}\bullet, \bullet{-}\bullet{-}\bullet\}$. So $\mathcal{B}$ has 1 object of size one and 2 objects of size three. The first few terms in the counting sequence for the class $\mathcal{A} = \mathrm{SEQ}(\mathcal{B})$ are $1, 1, 1, 3, 5, 7, 13, 23, \ldots$ Answer the questions below.

   (a) List the 5 elements of size four and the 7 elements of size five in $\mathcal{A}$.

   Solution:

   We list the 5 elements of size 4:

$$(\bullet, \bullet, \bullet, \bullet),\ (\bullet, \bullet\ \bullet{-}\bullet),\ (\bullet\ \bullet{-}\bullet, \bullet),\ (\bullet, \bullet{-}\bullet{-}\bullet),\ (\bullet{-}\bullet{-}\bullet, \bullet)$$

   Notice that the first element is an (ordered) 4-tuple, the second and third are ordered triples, and the last two are ordered pairs.

   (b) Find the generating function of $\mathcal{A}$.

   Solution:

   The ordinary generating function of $\mathcal{B}$ is $B(x) = x + 2x^3$. It follows that

$$A(x) = \frac{1}{1 - x - 2x^3}$$

   (c) Find the generating function of $\mathrm{SEQ}(\bullet{-}\bullet\mathcal{A})$. List all objects of size five.

   Solution:

   The ordinary generating function is

$$\frac{1}{1 - x^2 A(x)} = \frac{1}{1 - \frac{x^2}{1-x-2x^3}}$$

   The first 12 terms of the counting sequence of this class are 1, 0, 1, 1, 2, 5, 9, 18, 37, 73, 146, 293. As you can see, there should be 5 elements of size five and they are

$$(\bullet{-}\bullet, \bullet{-}\bullet{-}\bullet),\ (\bullet{-}\bullet, \bullet\ \bullet{-}\bullet),\ (\bullet{-}\bullet, \bullet{-}\bullet, \bullet),\ (\bullet{-}\bullet, \bullet, \bullet{-}\bullet),\ (\bullet{-}\bullet, \bullet, \bullet, \bullet)$$

   The first two elements are ordered pairs, the next two are ordered triples, and the last item is an ordered 4-tuple.

2. Let $N(x) = x(1-x)^{-2}$ and notice that the counting sequence is $\{n\}_{n\geq 0}$.
   (a) Let $\sum_n f_n x^n = E(x) = (1 - N(x))^{-1} - 1$ and find the first 6 terms of $\{e_n\}_n$.

   Solution:

   Using a CAS, the first 11 terms are 0, 1, 3, 8, 21, 55, 144, 377, 987, 2584, 6765.

   (b) The sequence above is actually the even numbered terms of a very famous sequence. Identify the sequence and prove your claim.

   Solution:

   These look like the even terms from the (shifted) Fibonacci sequence, 0, 1, 1, 2, 3, 5, 8, 13, 21. So let $F(x) = x(1-x-x^2)^{-1}$ (the ordinary generating of the shifted Fibonacci sequence). Then the even part of $F(x)$ is

$$\begin{aligned}
\frac{F(x)+F(-x)}{2} &= \frac{1}{2}\left(\frac{x}{1-x-x^2} + \frac{-x}{1+x-x^2}\right) \\
&= \frac{1}{2}\left(\frac{x+x^2-x^3-x+x^2+x^3}{(1-x-x^2)(1+x-x^2)}\right) \\
&= \frac{x^2}{1-3x^2+x^4} \\
&= E(x^2)
\end{aligned}$$

02/05 Let k be a fixed nonnegative integer and let L be a finite label set. Find the exponential generating functions of each of the following labeled structures.

Solution:

Most of these were done in class or in section 4.3 of Sagan's book here.

(a) $L \to B(L)$, set partitions on L.

(b) $L \to \left\{ {L \atop k} \right\}$, set partitions on L of size k.

(c) $L \to \left\{ {L \atop k} \right\}_o$, ordered set partitions on L of size k. *Note:* The blocks are ordered. So, for example, $12/3 \neq 3/12$.

(d) $L \to \mathfrak{S}(L)$, permutations on L.

(e) $L \to \left[{L \atop k} \right]$, permutations on L with exactly k cycles.

(f) $L \to \left[{L \atop k} \right]_o$, permutations on L with exactly k ordered cycles.

02/07

1. Show that $\left[{\cdot \atop 2}\right]_o = (\left[{\cdot \atop 1}\right]_o \times \left[{\cdot \atop 1}\right]_o)(\cdot)$.

   Solution:

   Done in class.

2. Show that $\left\{{\cdot \atop 2}\right\}_o = (\overline{E} \times \overline{E})(\cdot)$. More generally, show that $\left\{{\cdot \atop k}\right\}_o = \overline{E}^k(\cdot)$.

   Solution:

   Similar to problem 1 above.

02/07

1. Find the exponential generating function for $F_{\mathcal{S}}, F_{\mathcal{T}}$, and $F_{\mathcal{S}\times\mathcal{T}}$ for each of the following.
   (a) $\mathcal{S}(\cdot) = 2^{\cdot}$ and $\mathcal{T}(\cdot) = \left\{{\cdot \atop 2}\right\}$.

   Solution:

   In class we showed that $F_{\mathcal{S}}(x) = e^{2x}$ and $F_{\mathcal{T}}(x) = (e^x - 1)^2/2$. So by the Product Rule,

   $$F_{\mathcal{S}\times\mathcal{T}}(x) = \frac{e^{2x}(e^x-1)^2}{2}$$

   (b) $\mathcal{S}(\cdot) = 2^{\cdot}$ and $\mathcal{T}(\cdot) = \left[{\cdot \atop 3}\right]$.

   Solution:

   In class we showed that $F_{\mathcal{S}}(x) = e^{2x}$ and $F_{\mathcal{T}}(x) = \frac{1}{3!}\left(\ln\frac{1}{1-x}\right)^3$. So by the Product Rule,

   $$F_{\mathcal{S}\times\mathcal{T}}(x) = \frac{e^{2x}}{3!}\left(\ln\frac{1}{1-x}\right)^3$$

2. Show the following.
   (a) $\left\{{n \atop 2}\right\} = 2^{n-1} - 1$

   Solution:

   We appeal directly to the fact that $F_{\left\{{\cdot \atop 2}\right\}}(x) = \frac{(e^x-1)^2}{2}$. Thus

   $$\left\{{n \atop 2}\right\} = n![x^n]\frac{(e^x-1)^2}{2} = \frac{n!}{2}[x^n](e^{2x} - 2e^x + 1) = \frac{n!}{2}\left(\frac{2^n}{n!} - 2\frac{1}{n!}\right) = 2^{n-1} - 1$$

(b) $\begin{bmatrix} n+1 \\ 2 \end{bmatrix} = n! \sum_{k=1}^{n} \frac{1}{k}$

Solution:

According to Table 4.3.1

$$F_{\left[\substack{\cdot \\ 2}\right]}(x) = \frac{1}{2!}\left(\ln \frac{1}{1-x}\right)^2$$

Thus

$$\begin{aligned}
\begin{bmatrix} n+1 \\ 2 \end{bmatrix} &= (n+1)![x^{n+1}]F_{\left[\substack{\cdot \\ 2}\right]}(x) \\
&= (n+1)![x^{n+1}]\frac{1}{2!}\left(\ln \frac{1}{1-x}\right)^2 \\
&= \frac{(n+1)!}{2}[x^{n+1}]\left(\sum_{n\geq 1} \frac{x^n}{n}\right)^2 \\
&\overset{*}{=} \frac{(n+1)!}{2}[x^{n+1}]\sum_{n\geq 1}\sum_{k=1}^{n-1} \frac{1}{k}\,\frac{1}{n-k}\,x^n \\
&= \frac{(n+1)!}{2}\sum_{k=1}^{n} \frac{1}{k}\,\frac{1}{n+1-k} \\
&\overset{**}{=} \frac{(n+1)!}{2}\sum_{k=1}^{n} \frac{1}{n+1}\left(\frac{1}{k} + \frac{1}{n+1-k}\right) \\
&= \frac{n!}{2}\left(\sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{1}{k}\right) \\
&= n!\sum_{k=1}^{n} \frac{1}{k}
\end{aligned}$$

Here we used Wilf Rule 3 at step (*) and a partial fraction decomposition at step (**).

Solution:

Here's another approach. Let $c_n = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}$ and let $C(x) = \sum_n c_n\, x^n/n!$. It is routine (using either the Wilf Rules or the recursion for cycle numbers) to show that

$$C(x) = \frac{1}{1-x}\,\ln\frac{1}{1-x}$$

It follows that

$$\begin{aligned}\begin{bmatrix} n+1 \\ 2 \end{bmatrix} = n![x^n]C(x) &= n![x^n]\frac{1}{1-x}\,\ln\frac{1}{1-x} \\ &= n![x^n]\frac{1}{1-x}\sum_{n\geq 1}\frac{x^n}{n} \\ &\overset{(*)}{=} n![x^n]\sum_{n\geq 1}\sum_{k=1}^{n}\frac{1}{k}\,x^n \\ &= n!\sum_{k=1}^{n}\frac{1}{k}\end{aligned}$$

Here (*) follows by Wilf Rule 5 and we are done.

02/10

1. Show that $\mathfrak{S}([4]) = \Pi(c)([4])$. Here $c(\cdot) = \left[{\cdot \atop 1}\right]$.

   Solution:

   Outlined in class.

2. Find $\Pi(\left\{{\cdot \atop k}\right\})([5])$ for $k \in \{2, 3\}$.
3. Let j_n count the number of involutions in $\mathfrak{S}_n$ that have no fixed points. Give combinatorial proofs that $j_{2n+1} = 0$ and $j_{2n} = 1 \cdot 3 \cdot 5 \cdots (2n-1),\ n > 0$.

   Solution:

In exercise 01/13-2(b), we showed that involutions must be made up of 1-cycles and 2-cycles only. So if π is an involution with no fixed points, then π must only contain 2-cycles. In other words, $j_{2n+1} = 0$. Let $J([2n])$ be the collection of all involutions in $\mathfrak{S}([2n])$ with no fixed points.

Now let $n > 0$. Then there are $\binom{2n}{2}$ ways to choose two elements for the first cycle, followed by $\binom{2n-2}{2}$ to choose two elements for the second cycle, and so on. So by the product rule, there are $\binom{2n}{2} \cdot \binom{2n-2}{2} \cdots \binom{2}{2}$ ways to create an *ordered* involution with no fixed points. Since the order of the cycles is irrelevant, we have

$$\begin{aligned} j_{2n} = |J([2n])| &= \frac{1}{n!}\binom{2n}{2} \cdot \binom{2n-2}{2} \cdots \binom{2}{2} \\ &= \frac{2n(2n-1)\cdots 2 \cdot 1}{n!2^n} \\ &= \frac{2n(2n-2)\cdots 4 \cdot 2}{n!2^n}\,(2n-1)(2n-3)\cdots 5 \cdot 3 \cdot 1 \\ &= 1 \cdot 3 \cdot 5 \cdots (2n-1) \end{aligned}$$

as desired.

02/12 This a continuation of problem 3 from 02/10.

(a) Use the exponential formula to find the closed form of the exponential generating function $\sum_n j_n\, x^n/n!$. (C.f. 01/13-2(b) above)

Solution:

An involution is a permutation made up only of cycles of length 1 or 2. If fixed points are forbidden, then j_n must count only permutations made up of 2-cycles. So let $\mathcal{S}(L) = \begin{bmatrix} L \\ 1 \end{bmatrix}$ if $|L| = 2$ and $\mathcal{S}(L) = \emptyset$ otherwise. It follows that $s_n = \delta_2(n)$ and the exponential generating function of $\mathcal{S}(\cdot)$ is

$$F_{\mathcal{S}}(x) = \sum_n \delta_2(n)\,\frac{x^n}{n!} = \frac{x^2}{2}$$

It follows that

$$\begin{aligned} \sum_n j_n\,\frac{x^n}{n!} &= F_{\Pi(\mathcal{S})}(x) = e^{F_{\mathcal{S}}(x)} \\ &= e^{x^2/2} \end{aligned}$$

(b) Use the function from part (a) to give a generating function derivation of the formula in problem 3.

Solution:

Notice that

$$e^{x^2/2} = \sum_n 2^{-n} \frac{x^{2n}}{n!} \tag{10}$$

It is then easy to see that $j_{2n+1} = (2n+1)![x^{2n}]e^{x^2/2} = 0$ since the function in (10) is even. On the other hand,

$$\begin{aligned} j_{2n} &= (2n)!\,[x^{2n}] \sum_n 2^{-n} \frac{x^{2n}}{n!} \\ &= \frac{(2n)!}{n!} \frac{1}{2^n} \end{aligned}$$

which is equivalent to the formula given in problem 3 above.

(c) Let t_n count the number of permutations in $\mathfrak{S}_n$ with no fixed points whose cube is the identity. For example, let $\pi = (132) \in \mathfrak{S}_3$. Then π has no fixed points and $\pi^3 = \text{id}$. Find the closed form of the exponential generating function $\sum_n t_n\, x^n/n!$.

Solution:

Such a permutation must be made up only of cycles of length 3. So let $\mathcal{S}(L) = \left[{L \atop 1}\right]$ if $|L| = 3$ and $\mathcal{S}(L) = \emptyset$ otherwise. It follows that $s_n = 2\delta_3(n)$ and the exponential generating function of $F_{\mathcal{S}}(x) = 2x^3/3!$. It follows by the exponential formula that

$$\sum_n t_n \frac{x^n}{n!} = e^{2x^3/6}$$

Note: The reason that $s_3 = 2$ is because (132) and (123) are the only permutations in $\mathfrak{S}_3$ whose cube is the identity with no fixed points. We leave any remaining details to the student.

(d) What happens if we allow fixed points in part (c)?

Solution:

$$\sum_n t_n \frac{x^n}{n!} = e^{x+2x^3/6}$$

02/17

1. Let $f_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{1}{k}$. Answer the questions below.

   (a) Use binomial recursion to show that

$$f_n = \sum_{k=1}^{n} \frac{1}{k} \tag{11}$$

Solution:

We proceed by induction on n. Clearly $f_1 = 1$ in agreement with (11). Now suppose that (11) holds. Then

$$\begin{aligned}
f_{n+1} &= \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \frac{1}{k} \\
&\overset{(1)}{=} \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n}{k} \frac{1}{k} + \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n}{k-1} \frac{1}{k} \\
&\overset{(2)}{=} \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{1}{k} + \frac{1}{n+1} \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \\
&= f_n - \frac{1}{n+1} \left(\sum_{k=0}^{n+1} (-1)^{k} \binom{n+1}{k} - 1 \right) \\
&\overset{(3)}{=} \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{n+1} (0 - 1) \\
&= \sum_{k=1}^{n+1} \frac{1}{k}
\end{aligned}$$

as desired. Here we used the absorbtion/extraction property of the binomial coefficients on the right-hand sum in step (2). We also used binomial recursion in step (1) and Proposition 1 on the right-hand sum in step (3).

   (b) Let $g_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} f_k$. Use binomial inversion to conclude that $g_n = 1/n$.

Solution:

This is immediate.

   (c) Verify by direct calculation that $g_n = 1/n$.

2. Let $a_n = \sum_k (-1)^{n-k} \binom{n}{k} b_k$. Use a CAS and/or problem 1 from Quiz 5 to find the first 6 terms in each of the following sequences. Also, use the OEIS to determine whether any of these new sequences are "interesting".

   (a) b_n are the Bell numbers. That is, $b_n = 1, 1, 2, 5, 14, 42, 132, 429, \ldots$

   (b) b_n are the Motzkin numbers, $b_n = 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, \ldots$

   Note: The ordinary generating function for the Motzkin numbers is $\dfrac{1 - \sqrt{(1-x)^2 - 4x^2}}{2x^2}$.

(c) b_n are the Schröder numbers, $b_n = 1, 1, 3, 11, 45, 197, 903, 4279, \ldots$

Note: The ordinary generating function for the Schröder numbers is $\dfrac{1+x-\sqrt{1-6x+x^2}}{4x}$.

(d) b_n are the Riordan numbers, $b_n = 1, 0, 1, 1, 3, 6, 15, 36, 91, 232, 603, \ldots$

Note: The ordinary generating function for the Riordan numbers is $\dfrac{1+x-\sqrt{(1-x)^2-4x^2}}{2x(1+x)}$.

3. In class we mentioned that the transformation $F(x) \to \dfrac{1}{1-x}F\left(\dfrac{x}{1-x}\right)$ is called Euler's series transformation formula. Use this transformation formula to prove that

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k \quad \text{iff} \quad b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k \tag{12}$$

Hint: Mimic the transformation proof we did to prove the Stirling Inversion Theorem in class on Monday.

Solution:

Done in class.

4. On Quiz 5, we showed that the exponential generating function for the Lah numbers $\left\lfloor {n \atop k} \right\rfloor$ was

$$F_{\left\lfloor {\cdot \atop k} \right\rfloor}(x) = \frac{1}{k!}\left(\frac{x}{1-x}\right)^k \tag{13}$$

(a) Use (13) to discover a closed formula for $\left\lfloor {n \atop k} \right\rfloor$.

(b) Use (13) to show the following.

$$\left\lfloor {n+1 \atop k} \right\rfloor = \frac{n!}{(k-1)!}\sum_{j=0}^{n} \binom{j}{k-1} \tag{14}$$

Hint: Wilf Rule 1′ should help.

Solution:

Done in class.

**Exercises from the *A Walk Through Combinatorics*, 4^{th} *ed.*, Miklós Bóna, World Scientific