Throughout this exam, the instruction "find the generating function" always means find the closed form of the generating function.

1.) Let $\mathcal{B} = \{\bullet, \bullet\ \bullet\!\!-\!\!\bullet, \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\}$. So $\mathcal{B}$ has 1 object of size one and 2 objects of size three. The first few terms in the counting sequence for the class $\mathcal{A} = \text{SEQ}(\mathcal{B})$ are $1, 1, 1, 3, 5, 7, 13, 23, \ldots$
Answer the questions below. *Note:* To be clear, $\bullet\!\!-\!\!\bullet \notin \mathcal{A}$.

(a) (6 points) Notice that $(\bullet, \bullet\ \bullet\!\!-\!\!\bullet, \bullet) \in \mathcal{A}$. Now list the other 6 elements of size five in $\mathcal{A}$.

Solution:

The 6 other elements are

$$(\bullet, \bullet, \bullet, \bullet, \bullet)$$
$$(\bullet\ \bullet\!\!-\!\!\bullet, \bullet, \bullet),\ (\bullet, \bullet, \bullet\ \bullet\!\!-\!\!\bullet)$$
$$(\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet, \bullet, \bullet),\ (\bullet, \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet, \bullet),\ (\bullet, \bullet, \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet)$$

(b) (7 points) Find the generating function of $\mathcal{A}$.

Solution:

$$A(x) = \frac{1}{1 - x - 2x^3}$$

(c) (7 points) Find the generating function of $\mathcal{C} = \text{SEQ}(\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\,\mathcal{A})$.

Solution:

$$C(x) = \frac{1}{1 - x^3 A(x)} = \frac{1}{1 - \frac{x^3}{1-x-2x^3}} = \frac{1 - x - 2x^3}{1 - x - 3x^3}$$

2. (7 points) Show that

$$x^n = \sum_{k=0}^{n} \binom{n}{k} (1+x)^k (-1)^{n-k} \tag{1}$$

Solution:

According to the Binomial Theorem

$$(1+x)^n = \sum_k \binom{n}{k} x^k$$

Now (1) follows by inversion.

We can also prove this directly. We have

$$\begin{aligned}
\sum_k \binom{n}{k} (1+x)^k (-1)^{n-k} &\overset{*}{=} \sum_k \binom{n}{k} (-1)^{n-k} \sum_j \binom{k}{j} x^j \\
&= \sum_j x^j (-1)^n \sum_k \binom{n}{k} \binom{k}{j} (-1)^k \\
&\overset{**}{=} \sum_j x^j (-1)^n (-1)^n \delta_{nj} \\
&= x^n
\end{aligned}$$

as expected. Notice that we used the Binomial Theorem at step (*) and Proposition 1 at step (**).

3. (7 points) Let $A(x) \xleftrightarrow{\text{ogf}} \{a_n\}_n$ and $B(x) \xleftrightarrow{\text{ogf}} \{b_n\}_n$. Suppose that $a_n = \sum_k \binom{n}{k} \alpha^{n-k} b_k$ some nonzero real number α. Show that

$$A(x) = \frac{1}{1-\alpha x} B\left(\frac{x}{1-\alpha x}\right) \tag{2}$$

Solution:

$$
\begin{aligned}
A(x) = \sum_{n\geq 0} a_n x^n &= \sum_{n\geq 0} \sum_{k} \binom{n}{k} \alpha^{n-k} b_k\, x^n \\
&= \sum_{k\geq 0} b_k \alpha^{-k} \sum_{n\geq 0} \binom{n}{k} (\alpha x)^n \\
&= \sum_{k\geq 0} b_k \alpha^{-k} \frac{\alpha^k x^k}{(1-\alpha x)^{k+1}} \\
&= \frac{1}{1-\alpha x} \sum_{k\geq 0} b_k \left(\frac{x}{1-\alpha x}\right)^k \\
&= \frac{1}{1-\alpha x} B\left(\frac{x}{1-\alpha x}\right)
\end{aligned}
$$

as desired.

4. (20 points) Let $\mathcal{S}(\cdot) = \binom{\cdot}{1}$. Answer the questions below.

   (a) Find the exponential generating function $F_{\mathcal{S}}(x) = \sum_{n\geq 0} \binom{n}{1} x^n/n!$.

   Solution:

   This one is straightforward.

$$F_{\mathcal{S}}(x) = \sum_{n\geq 0} n\,\frac{x^n}{n!} = xD_x(e^x) = xe^x$$

   (b) List the distinct elements in $(\mathcal{S}\times\mathcal{S})([3])$. *Note:* These elements should be written as ordered pairs.

   Solution:

$$(1,2),\ (1,3),\ (2,1),\ (2,3),\ (3,1),\ (3,2)$$

   (c) Find the exponential generating function $F_{\mathcal{S}\times\mathcal{S}}(x)$. *Note:* $F_{\mathcal{S}\times\mathcal{S}}(x) \neq F_{\binom{\cdot}{2}}(x)$.

   Solution:

   By the Product Rule,

$$F_{\mathcal{S}\times\mathcal{S}}(x) = x^2e^{2x}$$

   (d) Use the exponential formula to find the exponential generating function for the *partition structure* $\Pi(\mathcal{S})$. In other words, find $F_{\Pi(\mathcal{S})}(x)$. *Note:* $\mathcal{S} = \overline{\mathcal{S}}$

   Solution:

   By the exponential formula

$$F_{\Pi(\mathcal{S})}(x) = e^{F_{\overline{\mathcal{S}}}(x)} = e^{xe^x}$$

   The first few terms of the counting sequence are

$$1, 1, 3, 10, 41, 196, 1057, 6322, 41393, 293608, 2237921, 18210094$$

5. (16 points) Let n and k be integers. Let $\left\lfloor {[n] \atop k} \right\rfloor$ be the collection of all partitions of $[n]$ into k linearly ordered blocks. As usual, let $\left\lfloor {0 \atop 0} \right\rfloor = 1$ and for $n > 0$, let $\left\lfloor {n \atop k} \right\rfloor = \left| \left\lfloor {[n] \atop k} \right\rfloor \right|$. As we noted in class, $\left\lfloor {n \atop k} \right\rfloor$ are called the Lah numbers (or Stirling Numbers of the 3rd kind). For example, $\left\lfloor {[3] \atop 2} \right\rfloor = \{12/3,\ 21/3,\ 13/2,\ 31/2,\ 23/1,\ 32/1\}$. It follows that $\left\lfloor {3 \atop 2} \right\rfloor = 6$. Notice that only the ordering within each block matters, not the order of the blocks themselves, so $32/1 = 1/32$, etc. It turns out that these numbers satisfy the following recursion.

$$\left\lfloor {n+1 \atop k} \right\rfloor = (n+k)\left\lfloor {n \atop k} \right\rfloor + \left\lfloor {n \atop k-1} \right\rfloor \tag{3}$$

together with additional boundary conditions $\left\lfloor {n \atop k} \right\rfloor = 0$ whenever $n < 0$ or $k \le 0$ or $k > n$.

(a) Find a combinatorial proof of the recursion (3).

Solution:

The left-hand side counts the number of partitions of $[n+1]$ into k linearly ordered blocks. Throughout the remainder of this proof, a partition means a partition with linearly ordered blocks.

Now for any partition in $\left\lfloor {[n+1] \atop k} \right\rfloor$, $n+1$ is either alone in a block or it is not.
In the first case, we can append $n+1$ to any of the partitions in $\left\lfloor {[n] \atop k-1} \right\rfloor$ to create a partition in $\left\lfloor {[n+1] \atop k} \right\rfloor$. Clearly there are $\left\lfloor {n \atop k-1} \right\rfloor$ ways to do this.

Otherwise, we can choose $\lambda \in \left\lfloor {[n] \atop k} \right\rfloor$, say $\lambda = B_1/B_2/\cdots/B_k$. Now we can place $n+1$ at the beginning of any block, e.g., $\lambda^j = B_1/B_2/\cdots/(n+1)B_j/\cdots/B_k$, so there are $k\left\lfloor {n \atop k} \right\rfloor$ ways to do this. Or we can place $n+1$ after any element (within any block), so there must be $n\left\lfloor {n \atop k} \right\rfloor$ ways to do this.

Since the 3 cases are distinct, we have shown that

$$\left\lfloor {n+1 \atop k} \right\rfloor = \left\lfloor {n \atop k-1} \right\rfloor + k\left\lfloor {n \atop k} \right\rfloor + n\left\lfloor {n \atop k} \right\rfloor$$

which is (3).

(b) Let $F(x) = F_{\left\lfloor \cdot \atop k \right\rfloor}(x) = \sum_{n\geq 0} \left\lfloor n \atop k \right\rfloor \frac{x^n}{n!}$. On Quiz 5 we showed that

$$F(x) = \frac{1}{k!}\left(\frac{x}{1-x}\right)^k \tag{4}$$

Use (4) to show that

$$\left\lfloor n \atop k \right\rfloor = \frac{n!}{k!}\binom{n-1}{k-1}$$

Solution:

This is straightforward.

$$\begin{aligned}
\left\lfloor n \atop k \right\rfloor &= \frac{n!}{k!}[x^n]\left(\frac{x}{1-x}\right)^k \\
&= \frac{n!}{k!}[x^n]\left(\frac{x}{1-x}\right)^k \frac{1-x}{1-x} \\
&= \frac{n!}{k!}[x^n]\left(\frac{x^k}{(1-x)^{k+1}} - x\,\frac{x^k}{(1-x)^{k+1}}\right) \\
&= \frac{n!}{k!}\left(\binom{n}{k} - \binom{n-1}{k}\right) \\
&= \frac{n!}{k!}\binom{n-1}{k-1}
\end{aligned}$$

6. (10 points) Let $\mathcal{S}(\cdot) = \binom{\cdot}{1}$. We saw in problem 4 above that $F_{\Pi(\mathcal{S})}(x) = e^{xe^x}$. Let

$$i_n = n![x^n]F_{\Pi(\mathcal{S})}(x) = n![x^n]e^{xe^x}$$

Find a sum formula for i_n. *Note:* The right-hand side of (5) below is an example of a sum formula.

Solution:

$$\begin{aligned}
\frac{i_n}{n!} &= [x^n]e^{xe^x} \\
&= [x^n]\sum_m x^m \frac{e^{mx}}{m!} \\
&= \sum_m \frac{1}{m!}[x^{n-m}]e^{mx} \\
&= \sum_m \frac{1}{m!}[x^{n-m}]\sum_k m^k \frac{x^k}{k!} \\
&= \sum_m \frac{1}{m!}\frac{m^{n-m}}{(n-m)!}
\end{aligned}$$

It follows that

$$\begin{aligned}
i_n &= \sum_m \frac{n!}{m!(n-m)!} m^{n-m} \\
&= \sum_m \binom{n}{m} m^{n-m}
\end{aligned}$$

It is worth mentioning that $i_0 = 1$ since $F_{\Pi(\mathcal{S})}(0) = 1$, but the sum formula above returns the indeterminate expression 0^0. In this case, we should specify that we define $0^0 = 1$.

7. (10 points) Let $l_0 = 1$ and for $n > 0$ let l_n count the number of ways to partition $[n]$ into an arbitrary number of nonempty linearly ordered blocks. In other words,

$$l_n = \sum_k \left\lfloor {n \atop k} \right\rfloor, \quad n \geq 0 \tag{5}$$

Prove that the closed form of the exponential generating function $L(x) = \sum_n l_n \, x^n/n!$ is

$$L(x) = e^{\frac{x}{1-x}} \tag{6}$$

Note: For this problem you may freely use my posted lecture notes or any of the other references listed on my Math 482 pages, but please do not use OEIS.

Solution:

Let $\mathcal{S}(\cdot) = \left\lfloor {\cdot \atop 1} \right\rfloor$. As we saw in problem 4 above,

$$F_{\mathcal{S}}(x) = L_1(x) = \frac{x}{1-x} = F_{\overline{\mathcal{S}}}(x)$$

Notice that the last equality holds since $\overline{\mathcal{S}} = \mathcal{S}$.

So the problem is asking us to partition $[n]$ in all possible ways and to linearly order each block in all possible ways. In other words, this problem is describing the partition structure $\Pi(\mathcal{S})$. Thus

$$L(x) = \sum_n l_n \frac{x^n}{n!} = F_{\Pi(S)}(x)$$

It now follows by the exponential formula that

$$F_{\Pi(S)}(x) = e^{F_{\overline{\mathcal{S}}(x)}} = e^{\frac{x}{1-x}}$$

8. (10 points) Let $A(x) \overset{\text{ogf}}{\longleftrightarrow} \{a_n\}_n$ and $B(x) \overset{\text{ogf}}{\longleftrightarrow} \{b_n\}_n$. Suppose that $a_n = \sum_k \binom{n}{k} \alpha^{n-k} b_k$ some nonzero real number α. In problem 3, we showed that

$$A(x) = \frac{1}{1-\alpha x} B\left(\frac{x}{1-\alpha x}\right) \tag{7}$$

Use (7) to show that

$$b_n = \sum_k \binom{n}{k} (-\alpha)^{n-k} a_k$$

Solution:

Notice that (7) implies that

$$B(x) = \frac{1}{1+\alpha x} A\left(\frac{x}{1+\alpha x}\right)$$

Thus

$$\begin{aligned}
b_n = [x^n]B(x) &= [x^n]\frac{1}{1+\alpha x} A\left(\frac{x}{1+\alpha x}\right) \\
&= [x^n]\frac{1}{1+\alpha x} \sum_k a_k \left(\frac{x}{1+\alpha x}\right)^k \\
&= [x^n]\sum_k a_k \frac{x^k}{(1+\alpha x)^{k+1}} \\
&= [x^n]\sum_k \frac{a_k}{(-\alpha)^k} \frac{(-\alpha x)^k}{(1-(-\alpha x))^{k+1}} \\
&= [x^n]\sum_k \frac{a_k}{(-\alpha)^k} \sum_n \binom{n}{k}(-\alpha)^n x^n \\
&= [x^n]\sum_n \sum_k \binom{n}{k} a_k (-\alpha)^{n-k} x^n \\
&= \sum_k \binom{n}{k} a_k (-\alpha)^{n-k}
\end{aligned}$$

as desired.