1. (10 points) Show that

$$\sum_{n\geq 1} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)}$$

Note: So $|\mu(n)| = 1$ if n is not divisible by a square and 0 otherwise.

Solution:

First, suppose that $|\mu(n)|$ is multiplicative. Then by Theorem 2.6.1 (Wilf)

$$\begin{aligned}\sum_n \frac{|\mu(n)|}{n^s} &= \prod_p \left\{1 + \frac{|\mu(p)|}{p^s} + \frac{|\mu(p^2)|}{p^{2s}} + \cdots\right\} \\ &= \prod_p (1 + 1/p^s + 0 + 0 + \cdots) \\ &= \prod_p (1 + 1/p^s)\,\frac{1 - 1/p^s}{1 - 1/p^s} \\ &= \prod_p (1 - 1/p^{2s}) \prod_p \frac{1}{(1 - 1/p^s)} \\ &= \frac{1}{\zeta(2s)}\,\zeta(s)\end{aligned}$$

Now if $f(n)$ is multiplicative and if $(m, n) = 1$, then

$$|f(mn)| = |f(m)f(n)| = |f(m)|\,|f(n)|$$

That is, $|f(n)|$ is multiplicative. Hence the application of Theorem 2.6.1 above is valid and the result follows.

2. (10 points) Let $\sigma(n)$ be the sum of the divisors of n. For example, $\sigma(6) = 1 + 2 + 3 + 6$. Show that

$$\sum_{n \geq 1} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1) \tag{1}$$

The proof is not difficult, but it can be quite messy. Please work out your proof on scratch paper and work out all of the kinks before submitting an easy-to-read version below.

Solution:

By exercise 04/08.1, $\sigma(n)$ is multiplicative and we can apply Theorem 2.6.1 to prove (1), just as we did in problem 1. A far simpler way is to recognize that we may use Rule $1''$. We have

$$\sigma(n) = \sum_{d|n} d \cdot 1$$

so by Rule $1''$

$$\begin{aligned}\sum_n \frac{\sigma(n)}{n^s} &= \sum_n \frac{n}{n^s} \sum_n \frac{1}{n^s} \\ &= \sum_n \frac{1}{n^{s-1}} \zeta(s) \\ &= \zeta(s-1)\zeta(s)\end{aligned}$$