1. (8 points) Let P_j be the poset of all positive divisors of $2 \cdot 3^j$ and call $x \le y$ if $x \mid y$. Find a simple formula for $\mu_{P_j}(x, y)$.

Solution:

In class we stated that $[x, y] \cong [1, y/x]$. It follows that

$$\mu_{P_j}(x,y) = \begin{cases} 1, & \text{if } y/x \in \{1, 6\} \\ -1, & \text{if } y/x \in \{2, 3\} \\ 0, & \text{otherwise.} \end{cases}$$

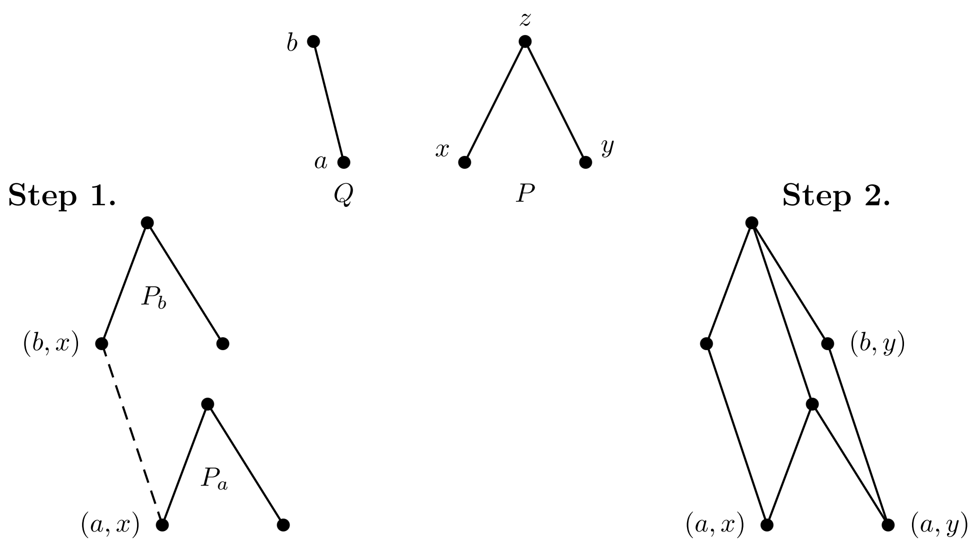

Figure 1: Product of posets, $Q \times P$.

To construct the Hasse diagram for $Q \times P$, complete the following steps (see Fig. 1).

Step 1. Replace the vertices of Q with identical (isomorphic) copies of P, labeled P_a and P_b in Fig. 1. For example, notice that $(a, x) < (b, x)$ in agreement with partial Hasse diagram shown in Step 1.

Step 2. Complete the Hasse diagram making sure that the covering relations are correct. For example, $(a, y) < (b, y)$ since $a \le b$ and $y \le y$. Also, (a, x) is not comparable with (b, y). Both of these relationships are correctly shown in Figure 1.

2. Let m and n be distinct positive integers and let D_m and D_n be the corresponding divisor lattices.

 (a) (8 points) Under what condition(s) is $D_m \times D_n \cong D_{mn}$? Justify your claim.

 Solution:

 We claim that $D_m \times D_n \cong D_{mn}$ whenever m and n are relatively prime, i.e., $(m, n) = 1$.

 Let $\phi : D_m \times D_n \to D_{mn}$ be defined by $\phi(r, s) = rs$. The map is clearly surjective for if $w \in D_{mn}$, then $w \mid mn$ and we can write $w = ab$ where $a \mid m$ and $b \mid n$, so that $w = \phi(a, b)$. Now let $\tau(n) = |D_n| = \sum_{d|n} 1$ count the number of divisors of n. (In class, we denoted this function by $d(n)$, but τ is the more frequently used symbol.) Now by exercise 04/01.1, τ is multiplicative. Thus

 $$|D_m \times D_n| = |D_m| \cdot |D_n| = \tau(m)\tau(n) = \tau(mn) = |D_{mn}|$$

 In other words, ϕ is a surjective map between two finite posets of the same size, so it must be a bijection. We leave it as an easy exercise to show that if $(a, b) \leq (x, y)$, then $ab \leq xy$.

 It now follows that $D_m \times D_n \cong D_{mn}$ whenever $(m, n) = 1$.

 (b) (4 points) Show by example that $D_m \times D_n \not\cong D_{mn}$ when the above condition(s) are not met.

 Solution:

 For example, $D_2 \times D_2 \cong B_2$ and has 4 elements. On the other hand, D_4 is a chain with only 3 elements, so the two are not isomorphic.