1. (10 points) Let $\mathcal{T} = \mathcal{T}^{\Omega}$, where $\Omega = \{0, 2, 3\}$. So $\mathcal{T}$ is the Ω-restricted class of plane trees such that each node has either 0, 2, or 3 children. The first few terms in the counting sequence for this class are 0, 1, 0, 1, 1, 2, 5, 8, 21, 42, 96, 222. *Note:* The size of each tree in $\mathcal{T}$ is measured by the number of nodes.

   (a) Sketch the two trees of size 5 and the five trees of size 6.

   (b) As usual, let $T(x)$ be the ordinary generating function for $\mathcal{T}$. Find the sum formula for $[x^n]T(x)$. *Hint:* What is the characteristic function for this class?

Solution:

Notice that $T(x)$ satisfies $T(x) = x\phi(T(x))$, with characteristic function $\phi(z) = 1 + z^2 + z^3$. So by the Lagrange Inversion formula,

$$\begin{aligned}
[x^n]T(x) &= \frac{1}{n}[z^{n-1}](1 + z^2 + z^3)^n \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} z^{2n-2k}(1+z)^{n-k} \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} \sum_j \binom{n-k}{j} z^{2n-2k+j} \\
&= \frac{1}{n}\sum_k \binom{n}{k}\binom{n-k}{2k-n-1}
\end{aligned}$$

One of the reasons that I include the first few terms in these sequence questions is so that students can check to see if their solution actually works for early terms in the sequence. For example,

$$\begin{aligned}
[x^5]T(x) &= \frac{1}{5}\sum_k \binom{5}{k}\binom{5-k}{2k-5-1} \\
&= \frac{1}{5}\left(0 + 0 + 0 + \binom{5}{3}\binom{5-3}{2(3)-6} + 0 + 0\right) \\
&= \frac{10}{5}
\end{aligned}$$

as expected.

2. (10 points) Let $\overline{\mathcal{T}} = \overline{\mathcal{T}}^{\Omega}$ where $\Omega = \{0, 2, 3\}$. However, this time we measure the size of each tree by the number of non-leaf nodes. Let $\overline{T}(x)$ be the ordinary generating function for $\overline{\mathcal{T}}$ and let $z(x) = \overline{T}(x) - 1$. One can show that $z = x\phi(z)$ for some characteristic function $\phi(z)$. Find $\phi(z)$.

   Hint: Such a tree is either $\circ$ (a node of size zero since it has no children) or $\mathcal{Z}_{\bullet} \times \overline{\mathcal{T}} \times \overline{\mathcal{T}}$ or $\mathcal{Z}_{\bullet} \times \overline{\mathcal{T}} \times \overline{\mathcal{T}} \times \overline{\mathcal{T}}$. Turn this into a class recursion and, once the recursion is established, take a look at Example 2 here.

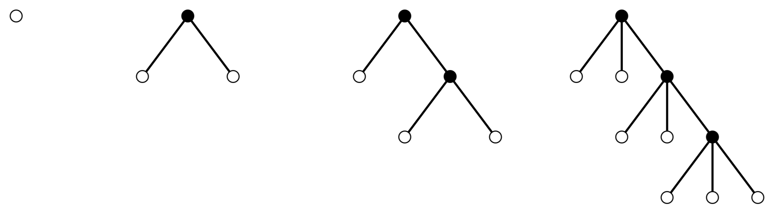

Figure 1: Trees of size 0, 1, 2, and 3.

Solution:

Figure 1 displays a few such trees. Nodes of weight zero are indicated using the symbol $\circ$. It turns out that $\phi(z) = (1+z)^2(2+z)$ so that

$$\begin{aligned}
[x^n]T(x) &= \frac{1}{n}[z^{n-1}](1+z)^{2n}(2+z)^n \\
&= \frac{1}{n}[z^{n-1}](2+5z+4z^2+z^3)^n \\
&= \frac{1}{n}\sum_{k}^{n}\binom{n}{k}\sum_{j}^{n-k}\binom{n-k}{j}\binom{n-k-j}{2n-3k-2j+1}2^k 5^j 4^{2n-3k-2j+1}
\end{aligned}$$

The first few terms of this sequence are

$1, 2, 10, 66, 498, 4066, 34970, 312066, 2862562, 26824386, 255680170, 2471150402, \ldots$

Here are the details. According to the hint,

$$\overline{\mathcal{T}} = \mathcal{Z}_{\circ} + \mathcal{Z}_{\bullet} \times \overline{\mathcal{T}} \times \overline{\mathcal{T}} + \mathcal{Z}_{\bullet} \times \overline{\mathcal{T}} \times \overline{\mathcal{T}} \times \overline{\mathcal{T}} \tag{1}$$

Notice that the ordinary generating functions of $\mathcal{Z}_{\circ}$ and $\mathcal{Z}_{\bullet}$ are x^0 and x^1, respectively. It follows that

$$\overline{T}(x) = 1 + x\overline{T}(x)^2 + x\overline{T}(x)^3 \tag{2}$$

Now let $z(x) = \overline{T}(x) - 1$, then

$$\begin{aligned}
z(x) &= x((1+z(x)^2) + (1+z(x)^3)) \\
&= x(2+5z(x)+4z(x)^2+z(x)^3)
\end{aligned}$$

It follows that

$$\phi(z) = (2+5z+4z^2+z^3) \tag{3}$$

and we are done.

Solution:

The original problem asked for a sum formula for $[x^n]\overline{T}(x)$, so let's derive such a formula. Let $W(z) = 1 + z$. Then $W'(z) = 1$ and by the Lagrange Inversion formula

$$\begin{aligned}
[x^n]T(x) = [x^n]W(z(x)) &= \frac{1}{n}[z^{n-1}]W'(z)\phi(z)^n \\
&= \frac{1}{n}[z^{n-1}](2 + 5z + 4z^2 + z^3)^n \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} 2^k z^{n-k}(5 + 4z + z^2)^{n-k} \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} 2^k z^{n-k} \sum_j \binom{n-k}{j} 5^j z^{n-k-j}(4+z)^{n-k-j} \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} 2^k z^{n-k} \sum_j \binom{n-k}{j} 5^j z^{n-k-j} \sum_l 4^l \binom{n-k-j}{l} z^{n-k-j-l} \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} \sum_j \binom{n-k}{j} \sum_l \binom{n-k-j}{l} 2^k 5^j 4^l z^{n-k} z^{n-k-j} z^{n-k-j-l} \\
&= \frac{1}{n}[z^{n-1}]\sum_k \binom{n}{k} \sum_j \binom{n-k}{j} \sum_l \binom{n-k-j}{l} 2^k 5^j 4^l z^{3n-3k-2j-l}
\end{aligned}$$

Now $3n - 3k - 2j - l = n - 1$ implies that $l = 2n - 3k - 2j + 1$, so that

$$[x^n]T(x) = \frac{1}{n}\sum_k \binom{n}{k} \sum_j \binom{n-k}{j} \binom{n-k-j}{2n-3k-2j+1} 2^k 5^j 4^{2n-3k-2j+1}$$

as desired.