1. (10 points) Let $\mathcal{S}(\cdot) = 2^{\cdot}$ and $\mathcal{T}(\cdot) = \left\{ {\cdot \atop 2} \right\}$. Let $s_n = |2^{[n]}| = 2^n$ and let $t_n = |\left\{ {[n] \atop 2} \right\}|$. According to a theorem that we discussed in class today,

$$|(\mathcal{S} \times \mathcal{T})([3])| = \sum_{k=0}^{3} \binom{3}{k} s_k t_{3-k} = \binom{3}{0} \cdot 1 \cdot 3 + \binom{3}{1} \cdot 2 \cdot 1 + 0 + 0 = 9 \tag{1}$$

List all 9 elements in $(\mathcal{S} \times \mathcal{T})([3])$.

Solution:

Looking at (1), we should find 3 ordered pairs of the form $(\emptyset, \cdot/\cdot)$ where $\cdot/\cdot \in \left\{ {[3] \atop 2} \right\}$. We should also have 6 ordered pairs of the form $(\emptyset|k, \cdot/\cdot)$. Here $a|b$ means either a or b and $\cdot/\cdot \in \left\{ {L \atop 2} \right\}$ with $L \subset [3]$ and $|L| = 2$. Thus

$$(\mathcal{S}\times\mathcal{T})([3]) = \underbrace{\{(\emptyset, 1/23), (\emptyset, 12/3), (\emptyset, 13/2)\}}_{\text{first form}} \cup \underbrace{\{(\emptyset, 2/3), (1, 2/3), (\emptyset, 1/3), (2, 1/3), (\emptyset, 1/2), (3, 1/2)\}}_{\text{second form}}$$

We can actually say a bit more about this example. According to the product rule for exponential generating functions, we must have

$$\begin{aligned} F_{\mathcal{S}\times\mathcal{T}}(x) &= F_{\mathcal{S}}(x) \cdot F_{\mathcal{T}}(x) \\ &= e^{2x} \cdot \frac{(e^x - 1)^2}{2!} \qquad\qquad \text{(See exercise 02/07 Problem 1)} \\ &= \frac{1}{2}(e^{4x} - 2e^{3x} + e^{2x}) \end{aligned}$$

It follows that

$$\begin{aligned} 3![x^3]F_{\mathcal{S}\times\mathcal{T}}(x) &= \frac{6}{2}[x^3](e^{4x} - 2e^{3x} + e^{2x}) \\ &= 3\left(\frac{4^3 - 2 \cdot 3^3 + 2^3}{3!}\right) \\ &= 9 \end{aligned}$$

in agreement with (1). The first 12 terms in the counting sequence of the exponential generating function $F_{\mathcal{S}\times\mathcal{T}}(x)$ are

$$0, 0, 1, 9, 55, 285, 1351, 6069, 26335, 111645, 465751, 1921029, \ldots$$