1. (10 points) A coach wishes to break up her n-member team into 3 practice squads. Each player on squad A will wear either a red or a blue jersey, those on squad B will wear yellow jerseys numbered from 1 to |squad B|, and squad C players will wear black jerseys and choose a squad captain. Let $t_0 = 0$ and for $n > 0$, let t_n count the number of ways that she can do this. Find the closed form of the exponential generating function $\sum_n t_n\, x^n/n!$. *Note:* This means that the squad B team is ordered and that squad C must have at least one player.

   Hint: The first few terms in this sequence are $0, 1, 8, 51, 312, \ldots$

   Solution:

   Let $i, j,$ and k be the number of players resp. on squad A, squad B, and squad C. Then

$$t_n = \sum_{i+j+k=n} \frac{n!}{i!j!k!} 2^i\, j!\, k \tag{1}$$

   So by the Wilf rules, we must have

$$\begin{aligned} T(x) = \sum_n t_n \frac{x^n}{n!} &= \sum_n 2^n \frac{x^n}{n!} \sum_n n! \frac{x^n}{n!} \sum_n n \frac{x^n}{n!} \\ &= e^{2x}\, \frac{1}{1-x}\, xe^x = \frac{xe^{3x}}{1-x} \end{aligned}$$

   Although it wasn't requested, we can see that

$$t_n = n![x^n] \frac{xe^{3x}}{1-x} = n![x^{n-1}] \frac{1}{1-x} e^{3x} = n! \sum_{k=0}^{n-1} \frac{3^k}{k!}$$

   So, for example,

$$t_3 = 3! \left(1 + \frac{3}{1} + \frac{3^2}{2!}\right) = 6(1 + 3 + 9/2) = 51, \text{ as expected}$$

   It is also worthwhile to list the 8 possibilities with two players, say 1 and 2. For example, $1^r|0|2$ indicates that player 1 is given a red jersey on squad A and player 2 would be the captain of squad C. Here are the other 7 lineups.

$$1^b|0|2,\ 2^r|0|1,\ 2^b|0|1$$
$$0|1|2,\ 0|2|1$$
$$0|0|1^c2,\ 0|0|12^c$$

   Hopefully, the notation is self-explanatory and it should be clear that these are the only cases.

Remark: (a) Notice that the right-hand side of (1) is zero whenever $k = 0$. That is, for any configuration that assigns zero players to squad C, the summand is 0, as expected.

(b) It is a worthwhile exercise to list the 51 possibilities for 3 players. This is something I plan to advocate for the entire semester - *List all possible configurations (within reason) for any exercise that you question.* Also, don't be afraid to use https://oeis.org/.

2. (a) (6 points) Use exercise 01/19 - #2 to show

$$\sum_{n,k\geq 0} \left\{ n \atop k \right\} t^k \frac{x^n}{n!} = e^{t(e^x-1)} \tag{2}$$

Solution:

Following the suggestion, we have

$$\begin{aligned}
\sum_{n\geq 0} \left\{ n \atop k \right\} \frac{x^n}{n!} &= \frac{1}{k!} \sum_{n\geq 0} k! \left\{ n \atop k \right\} \frac{x^n}{n!} \\
&= \frac{1}{k!} \sum_{n\geq 0} \sum_{j=0}^{k} \binom{k}{j} j^n (-1)^{j+k} \frac{x^n}{n!} \\
&= \frac{(-1)^k}{k!} \sum_j (-1)^j \binom{k}{j} \sum_{n\geq 0} \frac{(jx)^n}{n!} \\
&= \frac{(-1)^k}{k!} \sum_j (-1)^j \binom{k}{j} e^{xj} \\
&= \frac{(-1)^k}{k!} \sum_j \binom{k}{j} (-e^x)^j \\
&= \frac{(-1)^k}{k!} (1-e^x)^k \quad \text{(by the Binomial Theorem)}
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{n,k\geq 0} \left\{ n \atop k \right\} t^k \frac{x^n}{n!} &= \sum_{k\geq 0} t^k \sum_{n\geq 0} \left\{ n \atop k \right\} \frac{x^n}{n!} \\
&= \sum_{k\geq 0} \frac{t^k (e^x-1)^k}{k!} = e^{t(e^x-1)}
\end{aligned}$$

as desired.

(b) (4 points) Use (2) to rederive the exponential generating function for the Bell numbers. That is, let $\{b_n\}$ be the Bell numbers. Show that

$$\sum_{n\geq 0} b_n \frac{x^n}{n!} = e^{e^x-1} \tag{3}$$

Solution:

Recall that $b_n = \sum_{k=0}^{n} \left\{ n \atop k \right\}$. It follows that

$$\sum_{n\geq 0} b_n \frac{x^n}{n!} = \sum_{n\geq 0} \sum_{k=0}^{n} \left\{ n \atop k \right\} \frac{x^n}{n!} = \sum_{n\geq 0} \sum_{k=0}^{n} \left\{ n \atop k \right\} 1^k \frac{x^n}{n!} = e^{1(e^x-1)}$$