The Lagrange Inversion Formula - Applications (cont)

Recall that R. Stanley, et. al., have identified more than 200 combinatorial objects that can be counted by the Catalan numbers. We now introduce a similar family of numbers.

Suppose there are two candidates x and y running for the mayor of a small town. On election day, voters may choose candidate x, or candidate y, or "None of the Above(N)". If n votes are cast and the two candidates receive the same number of votes, in how many ways can the votes be tallied so that candidate x never trails candidate y. Let us refer to such a tally as a *legal* string. For example, the tallies xyN, xNy, Nxy, NNN are the only legal strings for 3 votes cast.

Now let $M([n])$ denote the set of all legal strings of length n (so n votes cast), and let $m_0 = 1$ and for $n > 0$, let $m_n = |M([n])|$. These are the Motzkin numbers and, like the Catalan numbers, there are numerous combinatorial objects that can be described using these numbers. The first few Motzkin numbers are $1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, \ldots$

Proposition 1. The Motzkin numbers m_n satisfy

$$m_n = m_{n-1} + \sum_{k=2}^{n} m_{k-2}m_{n-k}, \quad n > 0 \tag{1}$$

Proof: Let T be Motzkin string. Then T is a string that begins with an N or it begins with an x. In the first case, $T = NT'$ where $T' \in M([n-1])$ is a legal string. It follows that the number of such strings is m_{n-1}.

In the second case, let $k \geq 2$ denote the first occurrence of a y so that the prefix $xT''y$ is a legal string. Then $T = xT''yT'''$, where once again, T'' and T''' are (possibly empty) legal strings. Notice that $T'' \in M([k-2])$ and $T''' \in M([n-k])$. So by the product rule the number of such strings is $m_{k-2}m_{n-k}$. Summing k from 2 to n, we see the number of such strings is $\sum_{k=2}^{n} m_{k-2}m_{n-k}$ since each of these cases is clearly distinct.

The result now follows by the sum rule. □