The Lagrange Inversion Formula (cont)

Following Wilf we consider the following functional equation

$$z = x\phi(z) \tag{1}$$

Can we solve for z as an explicit function of x? Can we find a closed formula for the sequence of coefficients, $[x^n]z(x)$? *Note:* The functional equation (1) implies $z(0) = 0$.

Theorem 1 (The Lagrange inversion formula (LIF)). Suppose that $W(z)$ and $\phi(z)$ are formal power series in z with $\phi(0) \neq 0$. Then there is a unique formal power series $z = z(x) = \sum_n z_n x^n$, satisfying (1). In addition, the value of $W(z(x))$ when expanded in a power series in x about $x = 0$ satisfies

$$n[x^n]W(z(x)) = [z^{n-1}]\{W'(z)\phi^n(z)\} \tag{2}$$

The simplest version of the theorem occurs when we take $W(z) = z$. In that case, (2) reduces to

$$n[x^n]z(x) = [z^{n-1}]\phi^n(z) \tag{3}$$

There are numerous proofs in the literature. We present two.

Note: If $z(x)$ is a formal power series about $x = 0$, we follow the standard convention that $[x^m]z(x) = 0$ whenever $m < 0$.

Proof (First Proof of LIF): We proceed by induction on $n \geq 0$. For the base case, both sides of (2) are clearly 0 whenever $n = 0$. Now suppose that (2) is true for $0 < m < n$. As we attempted to illustrate when verifying binomial inversion using LIF, it is enough to show that the following holds for all k.

$$n[x^n]z^k(x) = k[z^{n-1}]\{z^{k-1}\phi^n(z)\} \tag{4}$$

We consider a few special cases:

(i) $k = 0$: The right-hand side of (4) is clearly 0, and $n > 0$ implies that $n[x^n]z^0(x) = n[x^n]\,1 = 0$.

(ii) $k > n$: Then $n - k < 0$ so that

$$n[x^n]z^k(x) \overset{(1)}{=} n[x^n]x^k\phi^k(z(x)) = n[x^{n-k}]\phi^k(z(x)) = 0$$

as we remarked above. The right-hand side is 0 for the same reason.

(iii) $k = n$: We have

$$\begin{aligned} n[x^n]z^n(x) &\overset{(1)}{=} n[x^n]x^n\phi^n(z(x)) = n[x^0]\phi^n(z(x)) = n\phi^n(z(0)) \\ &= n\phi^n(0) \\ &= n[z^0]\phi^n(z) = n[z^{n-1}]\{z^{n-1}\phi^n(z)\} \end{aligned}$$

Now suppose that $0 < k < n$. Then

$$\begin{aligned}
n[x^n]z^k(x) &\overset{(1)}{=} n[x^{n-k}]\phi^k(z(x)) \\
&\overset{(*)}{=} \frac{n}{n-k}[z^{n-k-1}]\left(\phi^k(z)\right)'\phi^{n-k}(z) \\
&= \frac{n}{n-k}[z^{n-k-1}]k\phi^{k-1}(z)\phi'(z)\phi^{n-k}(z) \\
&= \frac{k}{n-k}[z^{n-k}]nz\phi^{n-1}(z)\phi'(z) \\
&= \frac{k}{n-k}[z^{n-k}]zD_z\left(\phi^n(z)\right) \\
&\overset{(**)}{=} \frac{k}{n-k}[z^{n-k}](n-k)\phi^n(z) \\
&= k[z^{n-1}]z^{k-1}\phi^n(z)
\end{aligned}$$

as desired. Notice that we were able to use the induction hypothesis (4) at step (*) since $n-k < n$ and that we invoked Wilf Rule 2 at step (**). □

Remark. The induction proof is essentially the one presented in the 2014 paper by Surya and Warnke, *Lagrange Inversion Formula by Induction.*

Before introducing the second proof, we need to extend the ring of formal power series $\mathbb{C}[[x]]$ to the ring of formal Laurent series,

$$C((x)) = \left\{ \sum_{n \geq -N} c_n x^n \,\middle|\, N \in \mathbb{Z}, c_n \in \mathbb{C} \right\} \tag{5}$$

Notice that if $f(x) \in \mathbb{C}((x))$, then $\min\{n \,|\, [x^n]f(x) \neq 0, n \in \mathbb{Z}\} > -\infty$. Many of the rules from formal power series carry over directly to the ring of formal Laurent series, including sums, products, and the formal derivative. In addition, coefficient extraction works in $\mathbb{C}((x))$ just as it does in $\mathbb{C}[[x]]$. We also have the following

Lemma 2.

(i) If $z(x) \in \mathbb{C}((x))$ then $[x^{-1}]z'(x) = 0$.

(ii) If $f(x) \in \mathbb{C}[[x]]$ with $f(0) = 0$ and $f'(0) \neq 0$, then

$$[x^{-1}]f(x)^k f'(x) = \delta_{-1}(k) \tag{6}$$

for $k \in \mathbb{Z}$.

Proof: (i) is obvious. Let $f(x) = \sum_{n\geq 1} a_n\, x^n$, with $[x^1]f(x) = f'(0) \neq 0$. If $k = -1$ we have

$$\begin{aligned} f(x)^{-1}f'(x) &= \frac{a_1 + 2a_2 x + \cdots}{a_1 x + a_2 x^2 + \cdots} \\ &= \frac{1}{x} + \frac{a_2}{a_1} + \cdots \end{aligned}$$

so that

$$\begin{aligned} [x^{-1}]f(x)^{-1}f'(x) &= [x^{-1}]\frac{1}{x} + [x^{-1}]\left(\frac{a_2}{a_1} + \cdots\right) \\ &= 1 + 0 \end{aligned}$$

If $k \neq -1$ then

$$[x^{-1}]f(x)^k f'(x) = [x^{-1}]\frac{1}{k+1}(f(x)^{k+1})' = 0$$

by (i) above □

Remark. Notice that if $k < 0$, then $f(x)^k \notin C[[x]]$ since $[x^0]f(x) = f(0) = 0$. See Proposition 2 on the handout about formal power series. However, $f(x)^k \in C((x))$, the ring of formal Laurent series. We have more to say about this in the exercises.

Our second proof is lifted from the volume 2 of R. Stanley's *Enumerative Combinatorics*. We will show that although Stanley's version appears to be different, it is equivalent to the version presented above. We have

Theorem 3 (The Lagrange inversion formula). Let $F(x) = \sum_{n\geq 1} f_n\, x^n$ be a formal power series with $f_1 \neq 0$. Then

(7) $$n[x^n]F^{<-1>}(x)^k = k[x^{n-k}]\left(\frac{x}{F(x)}\right)^n, \quad k \in \mathbb{Z}$$

Here, $F^{<-1>}(x)$ is the compositional inverse of $F(x)$. That is, $F^{<-1>}(F(x)) = F(F^{<-1>}(x)) = x$.

Proof (Second proof of LIF): So let

(8) $$F^{<-1>}(x)^k = \sum_{j\geq k} p_j\, x^j$$

Then

$$x^k = \left\{F^{<-1>}(F(x))\right\}^k = \sum_{j\geq k} p_j\, F(x)^j$$

After differentiating both sides and dividing by $F(x)^n$ we obtain

(9) $$\frac{kx^{k-1}}{F(x)^n} = \sum_{j\geq k} j\, p_j\, F(x)^{j-n-1}F'(x)$$

Notice that we are treating both sides of (9) as Laurent series. For example,

$$\begin{aligned}\frac{kx^{k-1}}{F(x)^n} &= \frac{kx^{k-1}}{(f_1x + f_2x^2 + \cdots)^n} \\ &= kx^{k-n-1}(f_1 + f_2x + \cdots)^{-n}\end{aligned}$$

And the last expression is an element of $\mathbb{C}((x))$ since $(f_1 + f_2x + \cdots)^{-n} \in \mathbb{C}[[x]]$. Now by Lemma 2

$$[x^{-1}]F(x)^{j-n-1}F'(x) = \delta_{-1}(j - n - 1) = \delta_n(j) \tag{10}$$

so that

$$\begin{aligned}k[x^{n-k}]\left(\frac{x}{F(x)}\right)^n &= [x^{-1}]\frac{kx^{k-1}}{F(x)^n} \\ &\overset{(9)}{=} [x^{-1}]\sum_{j\geq k} j\, p_j\, F(x)^{j-n-1}F'(x) \\ &\overset{(10)}{=} \sum_{j\geq k} j\, p_j\, \delta_n(j) \\ &= np_n\end{aligned}$$

Thus

$$k[x^{n-k}]\left(\frac{x}{F(x)}\right)^n = np_n = n[x^n]\sum_{j\geq k} p_j\, x^j \overset{(8)}{=} n[x^n]F^{<-1>}(x)^k$$

as desired. □

To see that the two versions are equivalent, we let $z(x) = F^{<-1>}(x)$ and let $\phi(x) = x/F(x)$. Then $F(z(x)) = x$ and

$$x\phi(z(x)) = x\frac{z(x)}{F(z(x))} = x\frac{z(x)}{x}$$

so that (1) is satisfied. Making the appropriate substitutions in (7), we have

$$n[x^n]z(x)^k = k[x^{n-k}]\phi(x)^n = [x^{n-1}]kx^{k-1}\phi(x)^n \tag{11}$$

which is equivalent to (4), except that (11) holds for any $k \in \mathbb{Z}$.

Example 4. In his 2009 paper, B. Sagan answered a conjecture by J. Propp. Suppose the sequence $\{a_n\}_{n\geq 0}$ counts the number of a certain type of colored triangulations of an n-gon. Then

$$a_N = \begin{cases} \dfrac{2^n}{2n+1}\dbinom{3n}{n} & \text{if } N = 2n, \\[2ex] \dfrac{2^{n+1}}{2n+2}\dbinom{3n+1}{n} & \text{if } N = 2n+1. \end{cases} \tag{12}$$

Sagan's proof proceeded as follows

$$a_{2n+1} = \sum_{j=0}^{n} a_{2j} a_{2n-2j} \tag{13}$$

$$a_{2n} = \sum_{j=0}^{2n-1} a_j a_{2n-1-j} \tag{14}$$

Now we consider the ordinary generating functions

$$E(x) = \sum_{n\geq 1} a_{2n}\, x^n \quad \text{and} \quad O(x) = \sum_{n\geq 0} a_{2n+1}\, x^n$$

First observe that, for example,

$$\begin{aligned} a_{2(3)} &= \sum_{j=0}^{2(3)-1} a_j a_{2(3)-1-j} \\ &= a_0 a_5 + a_1 a_4 + \cdots + a_5 a_0 \\ &= 2 \sum_{j=0}^{3-1} a_j a_{2(3)-1-j} \end{aligned}$$

So, in general we have

$$a_{2n} = 2 \sum_{j=0}^{n-1} a_j a_{2n-1-j}$$

We leave it as an exercise to show that

$$2 \sum_{j=0}^{n-1} a_j a_{2n-1-j} = 2 \sum_{j=0}^{n-1} a_{2j+1} a_{2n-2j-2} = a_{2n} \tag{15}$$

Now

$$\begin{aligned} E(x) = \sum_{n\geq 1} a_{2n}\, x^n &\overset{(15)}{=} 2 \sum_{n\geq 1} \sum_{j=0}^{n-1} a_{2j+1} a_{2n-2j-2}\, x^n \\ &= 2x \sum_{n\geq 1} \sum_{j=0}^{n-1} a_{2j+1} a_{2n-2j-2}\, x^{n-1} \\ &= 2x \sum_{n\geq 0} \sum_{j=0}^{n} a_{2j+1} a_{2n-2j}\, x^n \\ &= 2x \sum_{n\geq 0} a_{2n+1}\, x^n \sum_{n\geq 0} a_{2n}\, x^n \\ &= 2xO(x)(1 + E(x)) \end{aligned}$$

That is,

$$E(x) = 2x(1 + E(x))O(x) \tag{16}$$

We leave it as an exercise to show

$$O(x) = (1 + E(x))^2 \tag{17}$$

Plugging (17) into (16) yields

$$E(x) = 2x(1 + E(x))^3 \tag{18}$$

This looks familiar (see Example 2 from Inversion Theorems - Part 2). So let $\phi(z) = 2(1 + z)^3$. Then $E(x) = x\phi(E(x))$ and we may invoke LIF. So by (3) we have

$$\begin{aligned}[x^n]E(x) &= \frac{1}{n}[z^{n-1}]\phi^n(z) \\ &= \frac{2^n}{n}[z^{n-1}](1+z)^{3n} \\ &= \frac{2^n}{n}\binom{3n}{n-1}\end{aligned}$$

which is equivalent to (12) when $N = 2n$. We leave the case $N = 2n + 1$ as an exercise.

Theorem 5 (Lagrange Inversion - Equivalent Forms). Suppose that $\phi(z)$ is a formal power series with $\phi(0) \neq 0$. Then there is a unique power series $f = f(x)$ such that

$$f(x) = x\phi(f(x)) \tag{19}$$

and for Laurent series $W(t)$ and $\psi(t)$ any $n \in \mathbb{Z}$ we have

$$[x^n]W(f) = \frac{1}{n}[t^{n-1}]W'(t)\phi(t)^n, \quad n \neq 0 \tag{20}$$

$$[x^n]W(f) = [t^n]\left(1 - \frac{t\phi'(t)}{\phi(t)}\right)W(t)\phi(t)^n \tag{21}$$

$$[x^n]\frac{\psi(f)}{1 - x\phi'(f)} = [t^n]\psi(t)\phi(t)^n \tag{22}$$

$$[x^n]\frac{\psi(f)}{1 - f\phi'(f)/\phi(f)} = [t^n]\psi(t)\phi(t)^n \tag{23}$$

We will first prove (19) and then (20) and show that the other formulas are equivalent to (20).

Proof: Since $\phi(0) \neq 0$, we have $1/\phi(t) \in \mathbb{C}[[t]]$ is unique and we may define $g(t) = t/\phi(t) \in \mathbb{C}[[t]]$. Notice that $g(0) = 0$ but $[t^1]g(t) \neq 0$ so that $f = f(x) = g^{<-1>}(x) \in \mathbb{C}[[x]]$ exists. It follows that

$$x = g(f(x)) = \frac{f(x)}{\phi(f(x))}$$

which is equivalent to (19). So using this notation, we may now state a fifth formula that is equivalent to the other four.

(24) $$[x^n]W(f) = [t^{n-1}]W(t)\frac{g'(t)}{g(t)}\left(\frac{t}{g(t)}\right)^n = [t^{-1}]\frac{W(t)g'(t)}{g(t)^{n+1}}$$

Now suppose that $W(t) = \sum_{k\geq N} w_k\, t^k$ for some $N \in \mathbb{Z}$. Then by (11) and linearity,

$$\begin{aligned}
[x^n]W(f(x)) = [x^n]\sum_{k\geq N} w_k\, f(x)^k &= \sum_{k\geq N} w_k\,[x^n]f(x)^k \\
&= \sum_{k\geq N} w_k\,\frac{1}{n}[t^{n-1}]kt^{k-1}\phi(t)^n \\
&= \frac{1}{n}[t^{n-1}]\sum_{k\geq N} k\, w_k\, t^{k-1}\phi(t)^n \\
&= \frac{1}{n}[t^{n-1}]W'(t)\phi(t)^n
\end{aligned}$$

and (20) is established. Once again let $g(t) = t/\phi(t)$. Then

$$D_t\left(\frac{W(t)}{g(t)^n}\right) = \frac{W'(t)}{g(t)^n} - \frac{nW(t)g'(t)}{g(t)^{n+1}}$$

Now by Lemma 2

$$[t^{-1}]D_t\left(\frac{W(t)}{g(t)^n}\right) = 0$$

so that

$$[t^{-1}]\frac{W'(t)}{g(t)^n} = n[t^{-1}]\frac{W(t)g'(t)}{g(t)^{n+1}}$$

It follows that

(25) $$\begin{aligned}
[t^n]\left(1 - \frac{t\phi'(t)}{\phi(t)}\right)W(t)\phi(t)^n &= [t^n]\frac{tg'(t)}{g(t)}W(t)\left(\frac{t}{g(t)}\right)^n \\
&= [t^n]t^{n+1}\frac{W(t)g'(t)}{g(t)^{n+1}} = [t^{-1}]\frac{W(t)g'(t)}{g(t)^{n+1}} \\
&= \frac{1}{n}[t^{-1}]\frac{W'(t)}{g(t)^n} = \frac{1}{n}[t^{-1}]W'(t)\frac{\phi(t)^n}{t^n} \\
&= \frac{1}{n}[t^{n-1}]W'(t)\phi(t)^n \\
&= [x^n]W(f(x))
\end{aligned}$$

which establishes (21) and (24).

It is easy to see that (22) and (23) are equivalent since $x = f(x)/\phi(f(x))$. We leave the remaining details as an exercise. □

Exercises

1. Let $f(x) = \sum_{n\geq 1} f_n\, x^n \in x\,\mathbb{C}[[x]]$. For any $g(x) \in \mathbb{C}((x))$, define the degree of $g(x)$ as we did for formal power series. That is,

$$\deg(g(x)) = \min\{n \in \mathbb{Z} \,|\, [x^n]g(x) \neq 0\}$$

   Let $k > 0$. Show that $f(x)^{-k} \in \mathbb{C}((x))$ with $\deg(f(x)^{-k}) = -k$. (Also, see Lemma 2.)

2. Verify identity (15).

3. Let $E(x)$ and $O(x)$ be as defined in Example 4. Show that $O(x) = (1 + E(x))^2$. Also, prove that

$$[x^n]O(x) = \frac{2^{n+1}}{2n+2}\binom{3n+1}{n}$$

4. In this exercise, we clean up a few of the missing details in Theorem 5

   (a) Show that

$$1 - \frac{t\phi'(t)}{\phi(t)} = \frac{tg'(t)}{g(t)}$$

   thus proving (25).

   (b) Now let $\psi(t) = W(t)\left(1 - \frac{t\phi'(t)}{\phi(t)}\right)$ and show that (21) and (23) are equivalent.