Binomial Inversion

Suppose that we have sequences $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$. In Math 481, we used the following orthogonal relationship

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m}(-1)^k = (-1)^n \delta_n(m) \tag{1}$$

to prove the Binomial Inversion Theorem. It states

$$f_n = \sum_{k=0}^{n} \binom{n}{k} g_k \quad \text{iff} \quad g_n = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} f_k \tag{2}$$

As an application, we used (2) to derive the following derangement formula,

$$!n = \sum_{k} (-1)^{n-k}\binom{n}{k} k! \tag{3}$$

Equation (3) is an example of something called a *binomial transform*, and I will refer to identities such as (1) as "orthogonality relationships", borrowing some terminology from linear algebra.

Over the next few lectures, we will explore other transforms, including the Stirling and Lah transforms, and we will examine the corresponding inversion formulas.

Stirling Inversion

We begin with

Proposition 1.

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{n-k} = \delta_n(m) \tag{4}$$

We outline one method of proof in the exercises. We will pursue a different tack below by using exponential generating functions to prove the analog of (2) for Stirling transforms. The orthogonality identity in (4) will then follow.

We have the following

Theorem 2. Suppose that we have sequences $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$. Then

$$f_n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} g_k \quad \text{iff} \quad g_n = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} f_k \tag{5}$$

In order to prove Theorem 2, we need the following identities from earlier in the semester.

$$\sum_n \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!} = \frac{1}{k!} \left(\ln \frac{1}{1-x} \right)^k \tag{6}$$

$$\sum_n \begin{Bmatrix} n \\ k \end{Bmatrix} \frac{x^n}{n!} = \frac{(e^x-1)^k}{k!} \tag{7}$$

Proof: Suppose the left-hand side of (5) holds and let

$$f(x) = \sum_n f_n \frac{x^n}{n!} \quad \text{and} \quad g(x) = \sum_n g_n \frac{x^n}{n!}$$

then

$$\begin{aligned} f(x) &= \sum_{n\geq 0} \frac{x^n}{n!} \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} g_k \\ &= \sum_{k\geq 0} g_k \sum_n \begin{Bmatrix} n \\ k \end{Bmatrix} \frac{x^n}{n!} \\ &= \sum_{k\geq 0} g_k \frac{(e^x-1)^k}{k!} && \text{(by (7))} \\ &= g(e^x - 1) \end{aligned}$$

In other words

$$f(x) = g(e^x - 1) \tag{8}$$

Now let $y = e^x - 1$ so that $x = \ln(1+y)$. Then (8) implies that

$$\begin{aligned} g(y) &= f(\ln(1+y)) \\ &= \sum_{k\geq 0} f_k \, \frac{\ln^k(1+y)}{k!} && \text{(by (6))} \\ &= \sum_{k\geq 0} f_k \sum_n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{y^n}{n!} \\ &= \sum_{n\geq 0} \sum_k (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} f_k \, \frac{y^n}{n!} \end{aligned}$$

It follows that

$$g_n = n![y^n]g(y) = n![y^n] \sum_{n\geq 0} \sum_k (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} f_k \, \frac{y^n}{n!} = \sum_k (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} f_k \tag{9}$$

as desired. The proof of the other direction is similar. $\square$