The Principle of Inclusion/Exclusion

We begin with an example.

Example 1. While doing his laundry last weekend, Tom discovered that many of his shirts were stained. Nine were splashed with vinegar, 12 had coffee stains, and 11 had pizza sauce on them (Tom is a messy eater). Eight shirts had both coffee and pizza stains, 4 had vinegar and pizza stains, and 4 had coffee and vinegar stains. If 2 of his shirts had all three stains and 4 had none at all, how many shirts were in Tom's laundry basket?

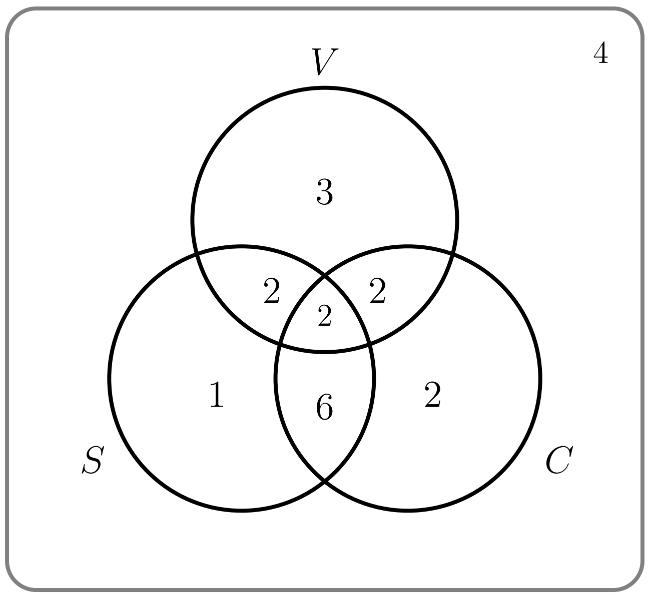

Figure 1: Tom's Dirty Laundry

A careful inspection of the diagram will show that we did not double count. Now let Q_0 equal the number of shirts in the basket that have no stains and let N equal the total number of shirts in the basket. Then

$$\begin{aligned} 4 = Q_0 &= N - (|V| + |C| + |S|) + (|VC| + |CS| + |SV|) - |VSC| \\ &= N - (9 + 12 + 11) + (4 + 8 + 4) - 2 = N - 18 \end{aligned} \tag{1}$$

So $N = 22$. (*Note:* Here we used the abbreviation $AB = A \cap B$ for sets A and B.)

The above example illustrates a very important technique in the theory of combinatorics. We have

Theorem 2. The Principle of Inclusion and Exclusion (PIE)

Suppose that we have a collection S of N objects where each object may satisfy one or more properties labeled $p_1, p_2, \ldots, p_r$. Now let $Q(p_i)$ denote the number of objects from S that have property p_i, $Q(p_i p_j)$ denote the number of objects that have properties p_i and p_j, and so on. Finally, let Q_0 denote the number of objects that have none of these properties. Then

$$Q_0 = N - \sum_i Q(p_i) + \sum_{i<j} Q(p_i p_j) - \sum_{i<j<k} Q(p_i p_j p_k) + \cdots + (-1)^m \sum_{j_1 < \cdots < j_m} Q(p_{j_1} \ldots p_{j_m}) + \cdots + (-1)^r Q(p_1 p_2 \ldots p_r) \qquad (2)$$

Proof: Let $x \in S$ and suppose that x satisfies exactly $m > 0$ properties. Then x contributes once to N, $\binom{m}{1}$ times to $\sum_i Q(p_i)$, $\binom{m}{2}$ times to $\sum_{i<j} Q(p_i p_j)$ and so on. In other words, x contributes

$$\binom{m}{0} - \binom{m}{1} + \binom{m}{2} + \cdots + (-1)^m \binom{m}{m} = 0$$

times to the right-hand side.

On the other hand, if x satisfies zero properties, then it is counted exactly once by N. □

Later we will give a more general proof that includes the above result as a special case.

Recall the derangement problem from lecture 9. It turns out that we can also study this problem using PIE. Once again, we let $!n = D_n$ equal the number of derangements (permutations with no fixed points) on $[n]$. Now let $S = \mathfrak{S}_n$ be our collection of objects. Then $N = n! = |S|$. Now let p_j denote the property that j remains fixed by a permutation from S. Then $Q(p_j)$ counts the number of permutations that fix j, $Q(p_jp_k)$ counts the number of permutations that fix j and k, and so on. The idea behind the Principle is to remove these unwanted permutations. It follows by (2) that

$$\begin{aligned}
D_n &= n! - \sum_i (n-1)! + \sum_{i<j}(n-2)! - \cdots + (-1)^m \sum_{i_1<\cdots<i_m}(n-m)! + \cdots + (-1)^n \\
&= \binom{n}{0}n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^m\binom{n}{m}(n-m)! + \cdots + \binom{n}{n}(-1)^n \\
&= \sum_{m=0}^{n}\binom{n}{m}(-1)^m \\
&= n!\sum_{m=0}^{n}\frac{(-1)^m}{m!}
\end{aligned}$$

In other words,

$$\frac{D_n}{n!} = \sum_{m=0}^{n}\frac{(-1)^m}{m!}$$

as we saw in lecture 9.

Example 3. Over the course of an m day school year, each of the n students in Ms. Baxter's class is chosen as leader for the day. Let $L(m,n)$ count the number of ways this can done so that every student gets to be class leader at least once. Show that

$$L(m,n) = \sum_{k=0}^{n}\binom{n}{k}(n-k)^m(-1)^k \tag{3}$$

Now let S be the set of **all** possible choices of class leaders over the semester. Notice that $N = n^m = |S|$ since S will include semesters where one or more students is never chosen as class leader. (*Note:* Many authors refer to this as over-counting.) Now number the students from 1 to n and let p_j be the property that the jth student is never selected as class leader. Notice that $Q(p_j) = (n-1)^m$ since we now have 1 less student to choose from. Similarly, $Q(p_j p_k) = (n-2)^m$ and so on. Then Q_0 is the counts number of semesters where every student was chosen to lead at least once. So by PIE we have

$$\begin{aligned}
L(m,n) = Q_0 = N - \sum_{j} Q(p_j) + \sum_{j<k} Q(p_j p_k) + \cdots \\
+ (-1)^{n-1} \sum_{j_1 < \cdots < j_{n-1}} Q(p_{j_1} \cdots p_{j_{n-1}}) \\
= n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m + \cdots \\
+ (-1)^r \binom{n}{r}(n-r)^m + \cdots + (-1)^{n-1}\binom{n}{n-1}(1)^m \\
= \sum_{r=0}^{n-1} \binom{n}{r}(n-r)^m(-1)^r = \sum_{r=0}^{n} \binom{n}{r}(n-r)^m(-1)^r
\end{aligned}$$

Recall that the Stirling numbers $\left\{ {n \atop k} \right\}$ counted the number of set partitions of $[n]$ into k blocks. The next example yields a "practical" formula for computing these numbers.

Example 4. For $n \geq m \geq 0$, show that

$$m!\left\{ {n \atop m} \right\} = \sum_{k=0}^{m} \binom{m}{k} k^n (-1)^{m-k} \tag{4}$$

Suppose that we have voters $v_1, v_2, \ldots, v_n$ and candidates $C_1, C_2, \ldots, C_m$.

Question. In how many ways can votes be cast so that each candidate receives at least one vote? Call each possible voting scenario an *election* result. Here we are actually concerned with who voted for each candidate, not just the candidate totals.

Answer 1. Notice that we can use ordered set partitions into exactly m blocks to identify who voted for each candidate, provided each candidate receives at least one vote. For example, if $n = 5$ and $m = 3$, then the ordered set partition 4/135/2 would indicate that v_4 voted for candidate C_1, voters v_1, v_3, and v_5 voted for candidate C_2 and finally, v_2 voted for C_3. Also notice that 135/4/2 represents a different election result. In other words, these are ordered set partitions.

So there are $\left\{ {n \atop m} \right\}$ ways to partition the set of voters into m nonempty blocks (so that each candidate receives some votes) and $m!$ ways to order these blocks. It follows by the product rule that there are $m!\left\{ {n \atop m} \right\}$ possible election results.

Answer 2. Here we let S be the set of **all** possible voting results. Since we allow for some candidates to receive zero votes, $|S| = m^n$. Now in a manner similar to the previous example, let p_j be the property that the jth candidate receives zero votes. Then $Q(p_j) = (m-1)^n$, $Q(p_j p_k) = (m-2)^n$ and so on. Now let Q_0 be the number of election results with none of these properties. Then Q_0 counts the number of possible election results where each candidate receives at least one vote and by PIE we have

$$\begin{aligned}
Q_0 &= N - \sum_{j} Q(p_j) + \sum_{j<k} Q(p_j p_k) + \cdots \\
&\qquad + (-1)^{m-1} \sum_{j_1 < \cdots < j_{n-1}} Q(p_{j_1} \cdots p_{j_{m-1}}) + (-1)^m Q(p_1 p_2 \cdots p_m) \\
&= m^n - \binom{m}{1}(m-1)^n + \binom{m}{2}(m-2)^n + \cdots + \binom{m}{m-1}(1)^n(-1)^{m-1} + 0 \\
&= \sum_{k=0}^{m} \binom{m}{k}(m-k)^n(-1)^k \\
&= \sum_{k=0}^{m} \binom{m}{m-k}(m-k)^n(-1)^k \\
&= \sum_{k=0}^{m} \binom{m}{k} k^n(-1)^{m-k}
\end{aligned}$$

and (4) is established.

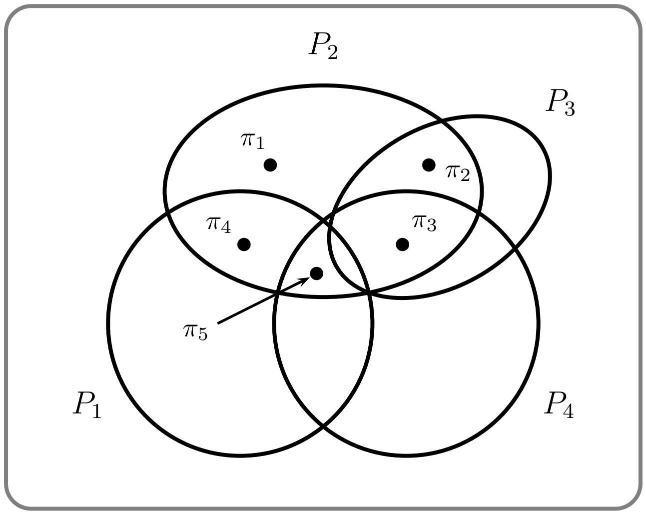

Figure 2: The Sieve Method

The Sieve Method

Let Γ be a set of objects and P be a set of properties that one or more of the objects possess. Let e_k denote the number of objects that have exactly k properties. How might we go about investigating this quantity?

Often it is easier to find out how many objects possess at least k properties. So let $S \subseteq P$ and let $N(\supseteq S)$ count the number of objects that have at least the properties in S. Now for a fixed $k \geq 0$ let

$$N_k = \sum_{|S|=k} N(\supseteq S) \tag{5}$$

Now let $\pi \in \Gamma$ and let $P(\pi)$ denote the subset of all of the properties that π possesses.

For example, in Figure 2 consider all of the subsets S of the properties $P = \{P_1, P_2, P_3, P_4\}$ with $|S| = 2$. Now $P(\pi_3) = \{P_2, P_3, P_4\}$ so that π_3 would contribute 3 times to N_2. Likewise, π_5 contributes 3 times to N_2. On the other hand, π_1 makes no contribution to N_k for $k > 1$ since $P(\pi_1) = \{P_2\}$.

Now

$$N_k = \sum_{|S|=k} N(\supseteq S) = \sum_{|S|=k} \sum_{\substack{\pi\in\Gamma \\ S\subseteq P(\pi)}} 1$$

$$= \sum_{\pi\in\Gamma} \sum_{\substack{|S|=k \\ S\subseteq P(\pi)}} 1 = \sum_{\pi\in\Gamma} \binom{|P(\pi)|}{k}$$

Now let $j \geq k$ and suppose that $|P(\pi)| = j$. How much does π contribute to N_k? It should be $\binom{j}{k}$. Thus

$$N_k = \sum_{j\geq k} \binom{j}{k} e_j$$

Now let $N(x)$ and $E(x)$ be the generating functions for N_k and e_j, respectively. Then

$$N(x) = \sum_k N_k x^k = \sum_k \sum_{j\geq k} \binom{j}{k} e_j x^k$$

$$= \sum_j e_j \left\{ \sum_k \binom{j}{k} x^k \right\} = \sum_j e_j (1+x)^j = E(x+1)$$

Since we are really interested in the e_j's, we usually write

$$E(x) = N(x-1) \tag{6}$$

Notice that the number of objects that have none of the given properties is

$$e_0 = E(0) = N(-1)$$

Example 5. Let's apply this analysis to Example 1. So $N = 22$ since there are 22 shirts. How many of the shirts have a coffee stain? How many have vinegar stains? How about pizza sauce stains. The answers are 12, 9, and 11, respectively. So $N_1 = 11 + 12 + 9 = 32$. Similarly, $N_2 = 16$ and $N_3 = 2$. It follows that

$$N(x) = 22 + 32x + 16x^2 + 2x^3$$

So by (6),

$$\begin{aligned} E(x) &= N(x-1) \\ &= 22 + 32(x-1) + 16(x-1)^2 + 2(x-1)^3 \\ &= 4 + 6x + 10x^2 + 2x^3 \end{aligned}$$

It is instructive to compare the following expression to (1).

$$E(0) = 22 + 32(-1)^1 + 16(-1)^2 + 2(-1)^3 = 4$$

Notice how quickly we were able to construct $N(x)$ and, ultimately, $E(x)$ from the given information. Now we can just read the results directly from the coefficients of $E(x)$. Remember, the coefficients of $E(x)$ give us exact information. So there are 4 shirts with exactly zero stains, 6 shirts that have exactly one stain and 10 shirts that have exactly two stains.

Now someone might point out that we're cheating here. In Example 1 we were asked to find N and we were given $e_0 = 4$. No problem.

$$N(x) = N + 32x + 16x^2 + 2x^3$$

So by (6),

$$\begin{aligned} E(x) &= N(x-1) \\ &= N + 32(x-1) + 16(x-1)^2 + 2(x-1)^3 \end{aligned}$$

Now

$$\begin{aligned} 4 = E(0) &= N(-1) \\ &= N - 32 + 16 - 2 \end{aligned}$$

so that $N = 22$ as we saw in Example 1.

We are now in a position to give another proof of Theorem 2.

Proof: Following the notation used in Theorem 2. Now for $1 \leq m \leq r$, we let $p_{j_1}, p_{j_2}, \ldots, p_{j_m}$ be a set of m properties. What can we say about $Q(p_{j_1}, p_{j_2}, \ldots, p_{j_m})$? It must count the number of objects in S that have at least those m properties. In other words,

$$N_m = \sum_{j_1 < \cdots < j_m} Q(p_{j_1} \ldots p_{j_m})$$

It follows that

$$N(x) = \sum_m N_m x^m$$

In particular,

$$\begin{aligned}
e_0 = N(-1) &= \sum_m N_m(-1)^m \\
&= \sum_m \sum_{j_1<\cdots<j_m} Q(p_{j_1}\dots p_{j_m})(-1)^m \\
&= N + \sum_{m=1}^{r}(-1)^m \sum_{j_1<\cdots<j_m} Q(p_{j_1}\dots p_{j_m}) \\
&= N + (-1)^1 \sum_i Q(p_i) + (-1)^2 \sum_{i<j} Q(p_i p_j) + \cdots \\
&\qquad + (-1)^m \sum_{j_1<\cdots<j_m} Q(p_{j_1}\dots p_{j_m}) + \cdots + (-1)^r Q(p_1 p_2 \dots p_r)
\end{aligned}$$

which is (2). $\square$

Example 6. How many permutations from $\mathfrak{S}_4$ fix exactly r elements?

There are $24 = 4!$ permutations. The 6 permutations that fix 1 are shown below.

$$(1)(2)(34),\ (1)(23)(4),\ (1)(234)$$
$$(1)(243),\ (1)(24)(3),\ e$$

Notice that some of these permutations fix other elements, but we don't care. So by symmetry, $N_1 = 24$. Now the following permutations fix 1 and 2.

$$e,\ (1)(2)(34)$$

So $N_2 = \binom{4}{2} \cdot 2 = 12$. In a similar fashion, $N_3 = \binom{4}{3} \cdot 1 = 4$. Finally, $N_4 = 1$ since the identity permutation is the only element in $\mathfrak{S}_4$ to fix $[4]$. It follows that

$$N(x) = 24 + 24x + 12x^2 + 4x^3 + x^4$$

so that

$$\begin{aligned} E(x) &= N(x-1) \\ &= 9 + 8x + 6x^2 + x^4 \end{aligned}$$

Notice that there are 9 permutations that fix zero objects (i.e., there are 9 derangements). There are 8 permutations that fix exactly one element and 6 permutations that fix exactly two elements.

Why are there no permutations that fix exactly three elements?

There is a Mathematica notebook that automatically generates $N(x)$, $E(x)$, and other useful information about $\mathfrak{S}_n$ for user specified values of n. It is available here: `https://tinyurl.com/ybqhvclb`.

Exercises

1. Let $L(m,n) = \sum_{k=0}^{n} \binom{n}{k} (n-k)^m (-1)^k$ as defined in Example 3. Show that $L(m,n) = 0$ whenever $0 < m < n$.