5.2 Set Partitions

Definition 1. Let $S = [n]$. We say the a collection of nonempty, pairwise disjoint subsets (called **blocks**) of S is a **set partition** if their union is S.

Example. Let $S = [4]$, then $\{1\}\{2,3,4\}$ is a partition of S into two subsets. Can you list the other 6?

$$\begin{array}{cc} \{1,2\} & \{3,4\} \\ \{1,3,4\} & \{2\} \\ \{1,2,3\} & \{4\} \\ \{1,4\} & \{2,3\} \\ \{1,2,4\} & \{3\} \\ \{1,3\} & \{2,4\} \end{array}$$

Definition 2. Now let $\left\{ {[n] \atop k} \right\}$ denote the collection of all partitions of $[n]$ into k subsets and let $S(n,k) = \left\{ {n \atop k} \right\}$ be the number of elements in $\left\{ {[n] \atop k} \right\}$. That is,

$$S(n,k) = \left\{ {n \atop k} \right\} = \left| \left\{ {[n] \atop k} \right\} \right|$$

These are called <u>Stirling numbers of the second kind</u> or <u>Stirling set numbers</u>. As we did with the binomial and multinomial coefficients, let's see if we can prove a recursion formula for these numbers.

Proposition 3. Let $n, k \in \mathbb{Z}$ and let $\left\{ {n \atop k} \right\} = 0$ whenever $k > n$ or $n < 0$ or $k < 0$. Finally, for $n > 0$ set $\left\{ {n \atop 0} \right\} = 0$. Then

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}, \quad ((n,k) \neq (0,0); \left\{ {0 \atop 0} \right\} = 1) \tag{1}$$

Proof: The exceptional cases are trivial, so we suppose that $1 \leq k \leq n$.

Question - In how many ways can we partition $[n]$ into k subsets?

LHS This is $\left\{ {n \atop k} \right\}$ by definition.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	b_n
0	1										1
1	0	1									1
2	0	1	1								2
3	0	1	3	1							5
4	0	1	7	6	1						15
5	0	1	15	25	10	1					52
6	0	1	31	90	65	15	1				203
7	0	1	63	301	350	140	21	1			877
8	0	1	127	966	1701	1050	266	28	1		4140
9	0	1	255	3025	7770	6951	2646	462	36	1	21147

Table 1: Stirling numbers of the second kind

RHS The set $\left\{ {[n] \atop k} \right\}$ contains partitions of two types. Some partitions contain the singleton subset $\{n\}$. The remaining partitions do not. When n is alone, then the remaining $n-1$ elements can be placed into $k-1$ subsets in $\left\{ {n-1 \atop k-1} \right\}$ ways. If n is not alone, we first partition $[n-1]$ into k subsets and then insert n into any of these subsets. So there are $k\left\{ {n-1 \atop k} \right\}$ ways to do this. Putting these together, there are $\left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}$ partitions of $[n]$ containing k subsets. $\square$

Table 1 lists the first 9 rows of the Stirling triangle. Do you notice any patterns in the table?

The sums of the entries in each row appear in the last column. They are the so-called **Bell numbers** and, by definition, give the number of ways to partition $[n]$ into nonempty blocks of any size. They will be discussed in more detail below.

Example 4. For a history class of n students, in how many ways can the students create m nonempty study groups? *Note:* Except for their members, the study groups are indistinguishable and not all students must participate.

We claim that the there are $\sum_k \binom{n}{k}\left\{{k \atop m}\right\}$ ways to do this. To see this notice that there $\binom{n}{k}$ to select a group of k students that will participate and $\left\{{k \atop m}\right\}$ ways to partition those k students into m nonempty study groups. So by the product rule, there are $\binom{n}{k}\left\{{k \atop m}\right\}$ ways to create the study groups with k students. Now if $j \neq k$ then the study groups created with j students and the study groups with k students are disjoint. So by the sum rule, there are $\sum_k \binom{n}{k}\left\{{k \atop m}\right\}$ ways to create m study groups in a class of n students.

For example, in a class of 4 students there are

$$\sum_{k=2}^{4} \binom{4}{k}\left\{{k \atop 2}\right\} = \binom{4}{2}\left\{{2 \atop 2}\right\} + \binom{4}{3}\left\{{3 \atop 2}\right\} + \binom{4}{4}\left\{{4 \atop 2}\right\}$$
$$= 6 \cdot 1 + 4 \cdot 3 + 1 \cdot 7 = 25$$

ways to create 2 nonempty study groups. One easily check that there are 140 ways to create 4 nonempty study groups in a class of 6 students. After looking in Table 1, we make the following conjecture.

$$\sum_k \binom{n}{k}\left\{{k \atop m}\right\} = \left\{{n+1 \atop m+1}\right\} \tag{2}$$

Proof: As we noted above, the left-hand side counts the number of m nonempty study groups from a class of n students. For the right-hand, we identify those who do not wish to participate as the students in the same block as $n+1$. □

Remark: Notice that we used a version of the "distinguished" element argument. In this case, we used it to identify which "study group" didn't exist. Notice that if $n+1$ appears alone in a block, then all students joined a study group.

Example 5. For $n \geq 0$ prove that

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}} \tag{3}$$

Proof: **Q.** How many ways can n students be assigned to x classrooms if rooms are allowed to remain empty?

Note: We remark that if there are 9 students and 3 classrooms, say A, B, C, we could use the ordered partition 236/17/4589 to indicate that students 2,3,6 were assigned to room A, students 1,7 to room B, etc. and the ordered partition 17/236/4589 would be a different assignment.

LHS. Clearly there are x^n ways to make such assignments.

RHS. Condition on the number of nonempty classrooms k. If there is only one nonempty room, then all of the students must be placed into the same room. Since there are x rooms, there are $x = \left\{ {n \atop 1} \right\} x^1$ ways to do this. For the general case, suppose that there are k nonempty rooms. Then there are $\left\{ {n \atop k} \right\}$ to divide the students into k subgroups and there are $x(x-1)\cdots(x-k+1) = x^{\underline{k}}$ ways to arrange the groups into the rooms. So by the product rule, there are $\left\{ {n \atop k} \right\} x^{\underline{k}}$ to distribute the students into k nonempty classrooms. Summing over all k produces the result.

Remark: Explain why (3) must hold for all real (complex?) numbers.

Set partitions can also be described using the *canonical form.*

Definition 6. Let $\sigma \in \left\{ {[n] \atop k} \right\}$, say $\sigma = B_1/B_2/\cdots/B_k$ written in standard block form. Now let $w(\sigma) = w_1 w_2 \cdots w_n \in [k]^n$ (an n-string on the alphabet $[k]$) defined by $w_i = j$ if and only if $i \in B_j$. Given a partition σ written in block form, we shall refer to $w(\sigma)$ as its canonical form.

For example, say $\sigma = 127/3/48/56 \in \left\{ {[8] \atop 4} \right\}$. Then $w(\sigma) = 11234413$ since, for example, $7 \in B_1$ iff $w_7 = 1$.

Question - Does the canonical form of a set partition give us any additional info about the size of Stirling set numbers?

Example 7. For $m, n \in \mathbb{N}$, show that

$$\sum_k \binom{n}{k} \left\{ {k \atop m} \right\} = \left\{ {n+1 \atop m+1} \right\}$$

Proof: We may assume that $0 \le m \le n$. Now, the right-hand side counts that number of ways to partition $[n+1]$ into $m+1$ blocks.

For the left-hand side, we condition on the block, call it B, containing $n+1$. If $n+1$ is alone, then there are $\binom{n}{n}\left\{ {n \atop m} \right\}$ ways to partition the remaining n elements across m blocks. If $|B| = 2$ then there are $\binom{n}{1} = \binom{n}{n-1}$ ways to choose the element that pairs with $n+1$ and $\left\{ {n-1 \atop m} \right\}$ ways to partition the remaining elements. So by the product rule, there are $\binom{n}{n-1}\left\{ {n-1 \atop m} \right\}$ to partition $[n+1]$ whenever $n+1$ is in a doubleton block. In general, there are $\binom{n}{k}$ ways to choose the $n-k$ elements that are paired with $n+1$ and $\left\{ {k \atop m} \right\}$ ways to partition the remaining k elements into m blocks and we apply the product rule to obtain $\binom{n}{k}\left\{ {k \atop m} \right\}$ to create $m+1$ blocks when $|B| = n-k+1$. Since each of these cases are disjoint, we can now sum on k and the result follows. $\square$

Example 8. For $n, m \in \mathbb{N}$ show that

$$\sum_{k=1}^{m} k\left\{ {n+k \atop k} \right\} = \left\{ {m+n+1 \atop m} \right\} \tag{4}$$

Hint: Use Proposition 3 and telescoping sums.

Proof: We give two proofs. Fix n and notice that by Proposition 3, the left-hand side can be rewritten as a telescoping sum. Let $a_k = \left\{ {n+k+1 \atop k} \right\}$. Then

$$\begin{aligned} k\left\{ {n+k \atop k} \right\} &= \left\{ {n+k+1 \atop k} \right\} - \left\{ {n+k \atop k-1} \right\} \\ &= a_k - a_{k-1} \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{k=1}^{m} k\left\{ {n+k \atop k} \right\} &= \sum_{k=1}^{m} a_k - a_{k-1} \\ &= a_m - a_0 \\ &= \left\{ {m+n+1 \atop m} \right\} - 0 \end{aligned}$$

as desired.

Now let's give a combinatorial proof. We'll let $m = 7$ and $n = 3$ to make the argument easier to follow. The right-hand side of (4) counts the number of ways to partition $[m+n+1] = [11]$ into $m = 7$ blocks.

For the left-hand side, we condition by identifying one or more singleton blocks and counting down. So what does $m\left\{ {n+m \atop m} \right\} = 7\left\{ {10 \atop 7} \right\}$ actually count? There are $\left\{ {10 \atop 7} \right\}$ ways to partition [10] into 7 blocks and we can place 11 into any of the blocks. So, by the product rule there are $7\left\{ {10 \atop 7} \right\}$ ways to do this. Observe that 11 is never in a singleton block.

Next, partition [9] into 6 blocks and let 11 occupy the last block. Now insert 10 into any of the first 6 blocks. By the product rule there are $6\left\{ {9 \atop 6} \right\}$ ways to do this. Notice that 11 is singleton, so this collection of partitions is disjoint from the previous collection.

Continuing, we partition [8] into 5 blocks, let 10 and 11 occupy the last two blocks, and insert 9 into any of the first 5 blocks. By the product rule there are

$5\left\{ {8 \atop 5} \right\}$ ways to do this. Notice that 10 and 11 are singleton, so this collection of partitions is disjoint from the previous collections.

We continue in this fashion until we partition [4] into 1 block, let 6–11 occupy the last 6 blocks, and insert 5 into the first block. Clearly, there are $1\left\{ {4 \atop 1} \right\}$ ways to do this. Once again, these partitions are disjoint from each of the collections above.

Now the result follows by the sum rule. □

Remark: Notice that there can never be more than 6 singleton blocks. Now one might argue that this method omits some partitions. It's pretty clear that if it does miss any, then singletons are involved. As an example, can you identify if and where the partition 1/2/34/58/679/10/11 was counted? What about 1/2/34/568/710/9/11 or 1/2/346/58/7/9/1011?

Proposition 9. The Bell numbers b_n satisfy the following recursion.

$$b_{n+1} = \sum_k \binom{n}{k} b_k, \quad n > 0,\ b_0 = 1 \tag{5}$$

Proof: We consider the number of set partitions of $[n+1]$. By definition, this is b_{n+1}. Now for each partition, we condition on the subsets that contain the number 1. If 1 is a singleton, there are b_n ways to partition the remaining n elements. Now suppose that 1 is in a doubleton. So there are $\binom{n}{1}$ to choose the element that is paired with 1, and there are b_{n-1} to partition the remaining $n-1$ elements. So by the product rule there $\binom{n}{1}b_{n-1}$ ways to partition $[n+1]$ in the case. It follows that for the general case, there $\binom{n}{k}b_{n-k}$ ways to partition $[n+1]$ whenever there are k elements in the same subset as 1. Since these cases are disjoint, we can sum over all values of k to obtain

$$b_{n+1} = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} = \sum_{k=0}^{n} \binom{n}{k} b_k \tag{6}$$

And the last equality follows by the symmetry of the binomial coefficients, $\binom{n}{k} = \binom{n}{n-k}$. □

The next example yields a "practical" formula for Stirling set numbers.

Example 10. For $n \geq m \geq 0$, show that

$$\sum_{k=0}^{m} \binom{m}{k} k^n (-1)^k = (-1)^m m! \left\{ {n \atop m} \right\} \tag{7}$$

Proof:

$$\begin{aligned} \sum_{k} \binom{m}{k} k^n (-1)^k &= \sum_{k=0}^{m} \binom{m}{k} \sum_{j} \left\{ {n \atop j} \right\} (k)_j (-1)^k \quad \text{(by (3))} \\ &= \sum_{j} j! \left\{ {n \atop j} \right\} \sum_{k} \binom{m}{k} \binom{k}{j} (-1)^k \\ &= \sum_{j} j! \left\{ {n \atop j} \right\} \delta_{jm} (-1)^m \\ &= (-1)^m m! \left\{ {n \atop m} \right\} \end{aligned}$$

□

Remark: We will revisit (7) in chapter 7 when we study sieve methods.