The Wilf Rules

Ordinary Generating Functions

Suppose that p is a polynomial, D is the usual derivative operator and

$$G(x) \quad \overset{\text{ogf}}{\longleftrightarrow} \quad \{g_n\}_{n\geq 0} \quad \text{and} \quad H(x) \quad \overset{\text{ogf}}{\longleftrightarrow} \quad \{h_n\}_{n\geq 0}$$

Then we have the following rules for ordinary generating functions.

Rule 1: $\dfrac{G(x) - g_0 - g_1 x - \cdots - g_{k-1}x^{k-1}}{x^k} \quad \overset{\text{ogf}}{\longleftrightarrow} \quad \{g_{n+k}\}_{n\geq 0}$

Rule 2: $p(xD)G(x) \quad \overset{\text{ogf}}{\longleftrightarrow} \quad \{p(n)g_n\}_{n\geq 0}$

Rule 3: $G(x)H(x) \quad \overset{\text{ogf}}{\longleftrightarrow} \quad \left\{\sum_{k=0}^{n} g_k h_{n-k}\right\}_{n\geq 0}$

Rule 4: $G(x)^k \quad \overset{\text{ogf}}{\longleftrightarrow} \quad \left\{\sum_{n_1+n_2+\cdots+n_k=n} g_{n_1}g_{n_2}\cdots g_{n_k}\right\}_{n\geq 0}$

Rule 5: $\dfrac{G(x)}{1-x} \quad \overset{\text{ogf}}{\longleftrightarrow} \quad \left\{\sum_{k=0}^{n} g_k\right\}_{n\geq 0}$

Exponential Generating Functions

Now let p and D be as defined above and

$$G(x) \quad \overset{\text{egf}}{\longleftrightarrow} \quad \{g_n\}_{n\geq 0} \quad \text{and} \quad H(x) \quad \overset{\text{egf}}{\longleftrightarrow} \quad \{h_n\}_{n\geq 0}$$

Then we have the following rules for exponential generating functions.

Rule 1′: $D^k G(x) \overset{\text{egf}}{\longleftrightarrow} \{g_{n+k}\}_{n\geq 0}$

Rule 2′: $p(xD)G(x) \quad \overset{\text{egf}}{\longleftrightarrow} \quad \{p(n)g_n\}_{n\geq 0}$

Rule 3′: $G(x)H(x) \quad \overset{\text{egf}}{\longleftrightarrow} \quad \left\{\sum_{k=0}^{n} \binom{n}{k} g_k h_{n-k}\right\}_{n\geq 0}$

Some Useful Identities from Math 481

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k} \qquad \text{(Multichoose)}$$

$$\binom{\alpha}{k} = (-1)^k \binom{k-\alpha-1}{k} \qquad \text{(Factor out negative)}$$

$$\sum_n \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}$$

Admissible Classes

Let $\mathcal{B}$ and $\mathcal{C}$ be classes. Then

Sum:	$\mathcal{A} = \mathcal{B} + \mathcal{C}$	$\Longrightarrow$	$A(x) = B(x) + C(x)$
Product:	$\mathcal{A} = \mathcal{B} \times \mathcal{C}$	$\Longrightarrow$	$A(x) = B(x)C(x)$
Sequence:	$\mathcal{A} = \text{SEQ}(\mathcal{B})$	$\Longrightarrow$	$A(x) = \frac{1}{1-B(x)}$
Powerset:	$\mathcal{A} = \text{PSET}(\mathcal{B})$	$\Longrightarrow$	$A(x) = \prod_{n\geq 1}(1+x^n)^{B_n} = \exp \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} B(x^k)$
Multiset:	$\mathcal{A} = \text{MSET}(\mathcal{B})$	$\Longrightarrow$	$A(x) = \prod_{n\geq 1}(1-x^n)^{-B_n} = \exp \sum_{k\geq 1} \frac{1}{k} B(x^k)$
Cycle:	$\mathcal{A} = \text{CYC}(\mathcal{B})$	$\Longrightarrow$	$A(x) = -\sum_{k\geq 1} \frac{\phi(k)}{k} \log(1 - B(x^k))$

where ϕ is the Euler totient function. For the last 4 constructions, we assume that $B(0) = 0$.

Labeled Structures

$S(L)$	s_n	egf
2^L	2^n	e^{2x}
$\left\{ L \atop k \right\}$	$\left\{ n \atop k \right\}$	$\frac{1}{k!}(e^x - 1)^k$
$\mathfrak{S}(L)$	$n!$	$\frac{1}{1-x}$
$\left[L \atop k \right]$	$\left[n \atop k \right]$	$\frac{1}{k!}\left(\ln \frac{1}{1-x}\right)^k$

Here $|L| = n$.