## Supplemental Material - Binomial Inversion

**Proposition 1.** Show that for $m, n \geq 0$

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m}(-1)^k = (-1)^n \delta_{nm} \tag{1}$$

where

$$\delta_{nm} = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{otherwise.} \end{cases}$$

*Proof:*

**Set 1.** Fix $n, m \geq 0$ let $E$ be the set of ordered pairs $(T, S)$ where $T \subseteq S \subseteq [n]$ for $|T| = m$ and $S$ contains an even number of elements. So if $|S| = k$ is even, how many elements (i.e., how many ordered pairs, $(T, S)$) does $E$ contain? Clearly there are $\binom{n}{k}\binom{k}{m}$ ways to select such ordered pairs from $[n]$ for each even $k$ and so

$$|E| = \sum_{k \text{ even}} \binom{n}{k}\binom{k}{m}$$

**Set 2.** In a similar manner we have

$$|O| = \sum_{k \text{ odd}} \binom{n}{k}\binom{k}{m}$$

Now if $n < m$ then both sets are empty. If $n = m$ is even, then $O$ is empty and $E$ contains a single element, namely $([n], [n])$. If $n = m$ is odd, then the reverse is true. In either case, the sets differ by one element and have the correct sign, in agreement with (1).

If $n > m$, then we need to show that $|E| = |O|$ since the right-hand side of (1) is 0 in this case. Now we define the following set operation. Let $x$ be the largest element in $[n]$ that is not in $T$ and let

□

$$S \triangle x = S \cup x \setminus S \cap x$$

be the symmetric difference of $S$ and $\{x\}$.

Now let $A = E \cup O$ and let $\iota : A \longrightarrow A$ be defined by $\iota((T, S)) = (T, S \triangle x)$. Notice that $\iota$ changes the parity of the set $S$. We leave it as an exercise to show that $\iota$ is an involution on $A$ with no fixed points. In particular, $\iota$ maps $E$ to $O$ and vice-versa. In other words, $|E| = |O|$.

*Remark:* If $n \geq m \geq 0$ then (1) can be rewritten as

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m}(-1)^{n-k} = \delta_{nm} \tag{2}$$

The above observations lead to the following interesting result.

**Theorem 2. Binomial Inversion** Suppose that we have sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$. Then

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k \quad \text{iff} \quad b_n = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} a_k \tag{3}$$

*Proof:* First we suppose that the right-hand equality holds in (3). Then

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} b_k &= \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} a_j \\
&= \sum_{j=0}^{n} (-1)^{-j} a_j \underbrace{\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j}(-1)^k}_{(-1)^n \delta_{nj} \text{ by (1)}} \\
&= \sum_{j=0}^{n} (-1)^{-j} a_j \, (-1)^n \delta_{nj} \\
&= a_n (-1)^{-n}(-1)^n \delta_{nn} = a_n
\end{aligned}$$

as expected. The proof of the other direction is similar. □

*Remark:* There is also a symmetric version of (3), namely

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k \quad \text{iff} \quad b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k \tag{4}$$

Let's see what we can do with this idea.

**Example 3.** Suppose that four people check their coats at a dinner party. Let's label the guests 1 through 4 and label their coats $A, B, C, D$, respectively. So coat $A$ belongs to 1, $B$ belongs to 2, etc. If the coats are randomly returned to the guests at the end of the evening, what is the probability that none of the guests retrieved their own coat?

Let's use the permutations of the word $ABCD$ to indicate how the coats were returned to the guests. For example, $ABCD$ indicates that each guest retrieved their own coat and $BACD$ indicates that only guests 3 and 4 got their coat. Now let $D(4,k)$ denote the number of ways that $k$ guests retrieve their own coat. Clearly, $D(4,4) = 1$. It is impossible for only 3 guests to get their own coat, so $D(4,3) = 0$. Now if we simply list all 24 possible permutations, we quickly determine that $D(4,2) = 6,\ D(4,1) = 8,$ and $D(4,0) = 9$. Notice that there are $4! = 24$ possible outcomes, as expected, and so the probability that none of the guests retrieves their own coat is $9/24 \approx 0.375$.

The above example is known as the *derangement* problem.

**Definition 4.** A permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ is called a <u>derangement</u> if $\pi_j \neq j$ for all $1 \leq j \leq n$. If $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ and $\sigma_j = j$ for some $j$, then $j$ is called a <u>fixed point</u> (of $\sigma$). So a derangement is a permutation with no fixed points.

Now let's try to answer the same question for an arbitrary number of $n$ guests. So let $D(n,k)$ count the number of ways that exactly $k$ of them retrieve their own coat. Notice then that $D(n,0)$ counts number of derangements of $n$ objects. If $k$ guests retrieve their own coat, then the remaining $n-k$ coats are deranged. Now there are $\binom{n}{k}$ ways to choose the points that are fixed and there are $D(n-k,0)$ ways to derange the remaining points, so by the product rule, there are $D(n,k) = \binom{n}{k}D(n-k,0)$ to fix exactly $k$ points. Since a permutation fixes 0 points, or 1 point, etc., it now follows by the sum rule that

$$\begin{aligned} n! &= \sum_k D(n,k) \\ &= \sum_k \binom{n}{k} D(n-k,0) \\ &= \sum_k \binom{n}{n-k} D(k,0) \\ &= \sum_k \binom{n}{k} D(k,0) \end{aligned}$$

Now by binomial inversion, we have our formula for derangements.

(5) $$!n =^{\text{def}} D(n,0) = \sum_k (-1)^{n-k} \binom{n}{k} k!$$

The symbol on the left is read as "$n$ subfactorial". Later in the semester we will re-derive (5) using the sieve method.

What happens as $n$ increases? Specifically, evaluate the following limit.

$$\begin{aligned} \lim_{n\to\infty} \frac{!n}{n!} &= \lim_{n\to\infty} \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k! \\ &= \lim_{n\to\infty} \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!(n-k)!} k! \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \\ &= \sum_{n\geq 0} \frac{(-1)^n}{n!} \\ &= 1/e \end{aligned}$$

So as $n$ increases, the probability that no guest retrieves their coat is roughly $1/e$, about 36.88%.

**Example 5.** Find a closed formula for

$$b_n = \sum_k \binom{n}{k} k 2^{k-1} (-1)^{n-k}$$

We try binomial inversion. According to (3), the equation above is equivalent to

$$\begin{aligned} n2^{n-1} &= \sum_k \binom{n}{k} b_k \\ &= \sum_k \frac{n}{k} \binom{n-1}{k-1} b_k \end{aligned}$$

It follows that

$$2^{n-1} = \sum_k \binom{n-1}{k-1} \frac{b_k}{k}$$

Now observe that if we set $b_k = k$, then

$$\begin{aligned} \sum_k \binom{n-1}{k-1} \frac{b_k}{k} &= \sum_k \binom{n-1}{k-1} \\ &= 2^{n-1} \end{aligned}$$

In other words, $b_n = n$.

**Definition 6.** Let $\{a_n\}_{n \geq 0}$ be a sequence and let $\{b_n\}_{n \geq 0}$ be the sequence defined by

$$b_n = \sum_k \binom{n}{k} a_k (-1)^k \tag{6}$$

The sequence $\{b_n\}$ is called the binomial transform of $\{a_n\}$. For example, $\{n\}$ is the binomial transform of $\{n2^{n-1}\}$ as we saw in the previous example.

**Example 7.** Find the closed form for the following sum.

$$\sum_k \binom{n}{k} (-1)^k \frac{2^k}{k+1}$$

After rearranging the extraction/absorbtion property $\binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}$, we have

$$\begin{aligned}
\sum_k \binom{n}{k} (-1)^k \frac{2^k}{k+1} &= \sum_k \frac{1}{k+1} \binom{n}{k} (-2)^{k+1} (-2)^{-1} \\
&= \frac{-1/2}{n+1} \sum_{k=0}^{n} \binom{n+1}{k+1} (-2)^{k+1} \\
&= \frac{-1/2}{n+1} \left[ \sum_{k=0}^{n+1} \binom{n+1}{k} (-2)^k - \binom{n+1}{0} (-2)^0 \right] \\
&= \frac{-1/2}{n+1} \left[ (1-2)^{n+1} - 1 \right] \\
&= \frac{1}{n+1} \frac{1-(-1)^{n+1}}{2} \\
&= \frac{1}{n+1} [n \text{ is even}]
\end{aligned}$$

## Exercises

1. Find the binomial transform (see (6)) of the sequences below.
   (a) $1, 1, 1, \ldots$
   (b) $\{n\}_{n\geq 0}$
   (c) $\{n^2\}_{n\geq 0}$
   (d) $\{\delta_{n,1}\}_{n\geq 0}$
   (e) $\{a^n\}_{n\geq 0},\ a \in \mathbb{R}$

2. Show that

$$a_n = \sum_{k=n}^{m} \binom{k}{n} b_k \quad \text{iff} \quad b_n = \sum_{k=n}^{m} \binom{k}{n} a_k (-1)^{n-k}$$

   *Note:* The index of summation is on the upper binomial index.

3. Prove the symmetric version of (3). Namely, prove that

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k \quad \text{iff} \quad b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k$$

4. Rewrite (3) as

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_{n-k}, \quad b_n = \sum_{k=0}^{n} \binom{n}{k} a_{n-k} (-1)^k$$

   and show that this implies

$$a_n = \sum_{k=0}^{n} \frac{b_{n-k}}{k!}, \quad b_n = \sum_{k=0}^{n} \frac{(-1)^k a_{n-k}}{k!}$$

   after appropriately redefining $a_n$ and $b_n$.

5. Find the closed form for the sum below.

$$\sum_{k \text{ even}} \binom{n}{k} \frac{(-1)^k}{k+1}$$