1. (8 points) How many lottery tickets are possible in following the modified version of MI47? Each ticket has 6 numbers between 1 and 47, but now a ticket can match any number at most three times. For example, $\{1,2,3,6,34,42\}, \{2,5,5,17,31,46\}$, and $\{4,4,19,19,19,36\}$ are valid tickets, but $\{6,6,12,12,12,12\}$ is not. *Express your answer as an integer and include the calculation.*

   Solution:

   In class we showed that the number of possible tickets when duplicates are permitted was

$$\sum_k \binom{47}{6-k}\binom{6-k}{k} = 19493673$$

   Now the number of possible tickets of the form $\{x,x,x,y,z,w\}$ is $\binom{47}{4}\binom{4}{1}$ since there are $\binom{47}{4}$ ways to choose 4 numbers and $\binom{4}{1}$ to select the number to be tripled. In a similar manner, there are $\binom{47}{2}\binom{2}{2}$ tickets of the form $\{x,x,x,y,y,y\}$. Finally, there are $\binom{47}{3}\binom{3}{1}\binom{2}{1}$ tickets of the form $\{x,x,x,y,y,z\}$. Since these cases as clearly distinct, we conclude that there are

$$19493673 + \binom{47}{4}\binom{4}{1} + \binom{47}{2}\binom{2}{2} + \binom{47}{3}\binom{3}{1}\binom{2}{1} = 20305504$$

   possible tickets.

2. (5 points) Today I went to the market to pick up a selection of apples, oranges, and peaches. Assuming the fruits within each of the three groups are indistinguishable, in how many ways can I return with 13 items? For example, I could have bought 5 apples and 8 peaches. *Express your answer as an integer and include the calculation.*

Solution:

This is a stars and bars argument. So there are 13 stars and 2 bars, hence the number of ways must be

$$\left(\!\!\binom{3}{13}\!\!\right) = \frac{15!}{2!13!} = 105$$

3. (7 points) Find the number of compositions of n into an even number of parts.

Solution:

If $n = 1$ then there are zero compositions with an even number of parts. If $n > 1$, then we claim that half of all compositions have an even number of parts. In other words, $2^{n-1}/2 = 2^{n-2}$. To see this recall that the number of compositions of n into k parts was given by $\binom{n-1}{k-1}$ and the total number of compositions is

$$2^{n-1} = \sum_k \binom{n-1}{k-1} \tag{1}$$

Now

$$\begin{aligned} 0 &= \sum_k \binom{n-1}{k-1}(-1)^k \qquad (2) \\ &= \sum_{\substack{k \\ k \text{ even}}} \binom{n-1}{k-1}(-1)^k + \sum_{\substack{k \\ k \text{ odd}}} \binom{n-1}{k-1}(-1)^k \\ &= \sum_{\substack{k \\ k \text{ even}}} \binom{n-1}{k-1} - \sum_{\substack{k \\ k \text{ odd}}} \binom{n-1}{k-1} \end{aligned}$$

Thus

$$\sum_{\substack{k \\ k \text{ even}}} \binom{n-1}{k-1} = \sum_{\substack{k \\ k \text{ odd}}} \binom{n-1}{k-1}$$

In other words, half of the compositions have an even number of parts. So by (1), we get 2^{n-2}.