1. (6 points) What is the largest coefficient in the expansion of $(x+y+z+w)^7$? EXPRESS YOUR ANSWER AS AN INTEGER.

   Solution:

   The coefficients have the form $\binom{7}{a,b,c,d}$ where $a+b+c+d=7$. After a bit of trial and error, it's easy to see that largest coefficient must be

$$\binom{7}{2,2,2,1} = 630$$

2. (7 points) Give a combinatorial proof of the identity below.

$$4^k = \sum_{j=0}^{k} 3^j \binom{k}{j}$$

   Solution:

   The left-hand side counts the number of k-words on the alphabet $\mathcal{A} = \{a, b, c, d\}$.

   For the right-hand side, there are $\binom{k}{k-j} = \binom{k}{j}$ ways to position $k-j$ occurrences of the letter a. Now the remaining positions may be filled with letters from the alphabet $\{b, c, d\}$ in 3^j ways. So by the product rule, there are $\binom{k}{j}3^j$ ways to create a k-word with exactly $k-j$ occurrences of a. Now we sum over j as j ranges from 0 to k since these collections are disjoint. The result follows.

3. (4 points) Find the sum below. JUSTIFY YOUR RESPONSE.

$$\sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k}$$

Solution:

By the Multinomial Theorem,

$$\begin{aligned} k^n &= \left(\sum_{j=1}^{k} 1\right)^n \\ &= \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} 1^{n_1} 1^{n_2} \cdots 1^{n_k} \\ &= \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} \end{aligned}$$

4. (3 points) How many terms are in the sum below? JUSTIFY YOUR RESPONSE.

$$\sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

For example, $\sum_{n_1+n_2=3} \binom{3}{n_1, n_2} x_1^{n_1} x_2^{n_2} = x_1^3 + 3x_1^2 x_2 + 3x_1 x_2^2 + x_2^3$ is a sum of 4 terms.

Solution:

There are as many terms as there are nonnegative integer solutions to the equation $n_1 + n_2 + \cdots + n_k = n$. But this is $\left(\!\binom{k}{n}\!\right)$, as we saw in chapter 3.