1. (10 points) A standard deck of 52 playing cards contains 4 suits $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$, with 13 cards in each suit. How many cards must be drawn (at random) to guarantee 3 cards from the same suit. *Justify your claim.*

   Solution:

   We must draw 9 cards. Let $r = 2$ and notice that we have $k = 4$ suits. If we draw only 8 cards, we could end up with two $\clubsuit$'s, two $\diamondsuit$'s, two $\heartsuit$'s, and two $\spadesuit$'s. On the other hand, since $9 > 2 \cdot 4$, we must have chosen at least 3 cards in a one suit, by the Pigeonhole Principle.

2. (10 points) Let $\mathfrak{J} \subset 2^{[n]} \times 2^{[n]}$ denote the set of all ordered pairs (A, B) with $A \cap B \neq \emptyset$. For example, for $n = 8$, we let $A = \{1, 2, 5\}$ and $B = \{1, 5\}$. Then $A \cap B = \{1, 5\}$ so that (A, B) and (B, A) are distinct elements in $\mathfrak{J}$. Find $|\mathfrak{J}|$.

   Solution:

   Let $n \in \mathbb{P}$. We claim that

$$
\begin{aligned}
|\mathfrak{J}| &= 4^n - \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = 4^n - \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (1)^k \\
&= 4^n - (1+2)^n \\
&= 4^n - 3^n
\end{aligned}
$$

   Here the second line follows from the Binomial Theorem.

   To justify the claim, observe that $|2^{[n]} \times 2^{[n]}| = 2^n \times 2^n$. Now we enumerate the ordered pairs $(A, B) \in \mathfrak{J}^c$ by indexing on $k = |A|$. If $k = 0$ then $A = \emptyset$ and B can be any subset of $[n]$, so there are $1 \cdot 2^n = \binom{n}{0} 2^n$ ordered pairs in this case. If $k = 1$ there are $\binom{n}{1}$ ways to choose $A = \{j\},\ j \in [n]$ and B can be any one of the 2^{n-1} subsets of $[n] \setminus \{j\}$. So by the product rule, there $\binom{n}{1} 2^{n-1}$ ordered pairs in $\mathfrak{J}^c$ when A is a singleton. It follows that the general term is of the form $\binom{n}{k} 2^{n-k}$, with $k = |A|$. Now since the indexed cases are mutually disjoint, we have

$$
|\mathfrak{J}^c| = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}
$$

   It now follows that

$$
\begin{aligned}
|\mathfrak{J}| &= 2^n \times 2^n - |\mathfrak{J}^c| \\
&= 4^n - \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}
\end{aligned}
$$

   as desired.