Exponential Generating Functions

The Wilf Rules for Exponential Generating Functions

Now suppose that $f \overset{\text{egf}}{\longleftrightarrow} \{a_n\}_{n\geq 0}$. What is the generating function for the sequence $\{a_{n+1}\}_{n\geq 0}$? We have

$$\sum_{n\geq 0} a_{n+1}\frac{x^n}{n!} = \sum_{n\geq 0} a_{n+1}(n+1)\frac{x^n}{(n+1)!} = f'$$

This gives us.

Rule $1'$**.** If $f \overset{\text{egf}}{\longleftrightarrow} \{a_n\}_{n\geq 0}$ and k is a positive integer, then

$$D^k f \overset{\text{egf}}{\longleftrightarrow} \{a_{n+k}\}_{n\geq 0} \tag{1}$$

Example 1. Consider the (shifted) Fibonacci numbers $F_0 = 0,\ F_n = f_{n-1},\ n > 0$ and let $G \overset{\text{egf}}{\longleftrightarrow} \{F_n\}_n$. It is easy to see that these numbers satisfy the recurrance

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0,\ F_1 = 1 \tag{2}$$

Now by Rule $1'$, G must satisfy the differential equation

$$G''(x) = G'(x) + G(x), \quad G(0) = 0,\ G'(0) = 1 \tag{3}$$

This equation can be solved by the method of characteristic polynomials to yield the solution

$$G(x) = \frac{e^{\phi x} - e^{-x/\phi}}{\sqrt{5}}, \quad \phi = \frac{1+\sqrt{5}}{2} \tag{4}$$

Now to find F_n we simply apply the operator $n![x^n]$ to (4). Thus

$$\begin{aligned} F_n = n![x^n]G(x) &= n![x^n]\frac{e^{\phi x} - e^{-x/\phi}}{\sqrt{5}} \\ &= \frac{n![x^n]}{\sqrt{5}}\sum_{n\geq 0}(\phi^n - (-1/\phi)^n)\frac{x^n}{n!} \\ &= \frac{1}{\sqrt{5}}\left(\phi^n - (-1/\phi)^n\right) \\ &= \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} \end{aligned}$$

It turns out that multiplying terms in the sequence by n is the same for egf as it was for ogf. We have

Rule $2'$**.** If $f \overset{\text{egf}}{\longleftrightarrow} \{a_n\}_{n\geq 0}$ and P is a polynomial, then

$$P(xD)f \overset{\text{egf}}{\longleftrightarrow} \{P(n)a_n\}_{n\geq 0} \tag{5}$$

The most interesting rule regarding egf's involves multiplication. Now suppose that $f \xleftrightarrow{\text{egf}} \{a_n\}_n$ and $g \xleftrightarrow{\text{egf}} \{b_n\}_n$, then by Rule 3 we have

$$\begin{aligned} fg &= \left\{\sum_{n\geq 0} \frac{a_n x^n}{n!}\right\}\left\{\sum_{m\geq 0} \frac{b_m x^m}{m!}\right\} \\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} \frac{a_k}{k!}\frac{b_{n-k}}{(n-k)!}\right)x^n \\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a_k b_{n-k}\right)\frac{x^n}{n!} \\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right)\frac{x^n}{n!} \end{aligned}$$

Rule $3'$**.** If $f \xleftrightarrow{\text{egf}} \{a_n\}_n$ and $g \xleftrightarrow{\text{egf}} \{b_n\}_n$, then

$$fg \xleftrightarrow{\text{egf}} \left\{\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right\}_n \tag{6}$$

Example 2. In an earlier lecture, we showed that the Bell numbers b_n satisfied the following recursion

$$b_{n+1} = \sum_k \binom{n}{k} b_k, \quad b_0 = 1, n > 0 \tag{7}$$

Now let $B \xleftrightarrow{\text{egf}} \{b_n\}_n$. We multiply both sides of by $x^n/n!$ sum over $n \geq 0$. By Rule $1'$, $B' \xleftrightarrow{\text{egf}} \{b_{n+1}\}_n$ and hence

$$\begin{aligned} B'(x) &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} \binom{n}{k} b_k\right)\frac{x^n}{n!} \\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} \binom{n}{k} b_k \cdot 1\right)\frac{x^n}{n!} \\ &= \left\{\sum_{n\geq 0} b_n \frac{x^n}{n!}\right\}\left\{\sum_{n\geq 0} \frac{x^n}{n!}\right\} \\ &= B(x)e^x \end{aligned}$$

by Rule $3'$. Now we can proceed as we did before. Rearranging, we have

$$\frac{B'(x)}{B(x)} = e^x \tag{8}$$

Integrating both sides yields

$$\ln B(x) = e^x + C = e^x - 1$$

since $B(0) = 1$. It follows that the exponential generating function for the Bell numbers is

$$B(x) = e^{e^x - 1}$$

as we have seen before.

Exponential Generating Functions

Example 3. A local soccer team has n players. The coach decides to split the players up into two groups. In the first group, players must choose either a green Jersey, a white Jersey, or a blue Jersey. The players in the second group must wear a yellow Jersey. In how many ways can the coach choose the two groups?

Let c_n count the number of ways that this can be done and let $C(x)$ be the corresponding egf. Now there $\binom{n}{k}$ to choose the k members for the first group and 3^k ways that the group members can choose their Jerseys. And there is only one way to choose the second group. So by the product rule, there $3^k\binom{n}{k} \cdot 1$ ways to choose a group of k students from a group of n students under the given conditions. Summing over all values of k yields

$$c_n = \sum_{k=0}^{n} \binom{n}{k} 3^k \tag{9}$$

It follows by Rule $3'$ that

$$\begin{aligned} C(x) &= \left\{ \sum_{n\geq 0} 3^n \frac{x^n}{n!} \right\} \left\{ \sum_{n\geq 0} \frac{x^n}{n!} \right\} \\ &= e^{3x} e^{x} \end{aligned}$$

Now

$$c_n = n![x^n]e^{4x} = 4^n$$

Now since (9) can be computed directly using the Binomial theorem, the last result isn't very impressive. Indeed,

$$c_n = \sum_{k=0}^{n} \binom{n}{k} 3^k = (1+3)^n$$

The next example is a bit more interesting.

Example 4. Rework the previous problem except as noted below.

a. This time the coach will choose a captain from the second group. In how many ways can this be done?

   Adapting the notation from Example 3, we see that for a fixed k, there are $3^k\binom{n}{k} \cdot (n-k)$ the coach can choose the groups. Once again, summing over all k yields

$$c_n = \sum_{k=0}^{n} \binom{n}{k} 3^k (n-k) \tag{10}$$

   It follows by Rule $3'$ that

$$\begin{aligned} C(x) &= \left\{ \sum_{n\geq 0} 3^n \frac{x^n}{n!} \right\} \left\{ \sum_{n\geq 0} n \frac{x^n}{n!} \right\} \\ &= e^{3x} x e^{x} \end{aligned}$$

   by Rule $2'$. Now

$$c_n = n![x^n]xe^{4x} = n![x^{n-1}]e^{4x} = n4^{n-1}$$

b. For this arrangement, the second group will be placed (ordered) in a line. In how many ways can this be done?

Again using the notation from Example 3, we see that for a fixed k, there are $3^k\binom{n}{k}\cdot(n-k)!$ ways that the coach can choose the groups. Summing over all k yields

$$c_n = \sum_{k=0}^{n}\binom{n}{k}3^k(n-k)! \tag{11}$$

It follows by Rule $3'$ that

$$\begin{aligned} C(x) &= \left\{\sum_{n\geq 0} 3^n\frac{x^n}{n!}\right\}\left\{\sum_{n\geq 0} n!\frac{x^n}{n!}\right\} \\ &= e^{3x}\frac{1}{1-x} \end{aligned}$$

Now

$$c_n = n![x^n]\frac{e^{3x}}{1-x}$$

The first 10 terms of the sequence are 1, 4, 17, 78, 393, 2208, 13977, 100026, 806769, 7280604.

A "closed form" was discovered for this sequence in 2017 by Peter Luschny, apparently. It turns out that

$$c_n = e^3\Gamma(1+n,3)$$

where Γ is the **upper incomplete gamma function** and is defined by the improper integral

$$\Gamma(s,x) = \int_x^{\infty} t^{s-1}e^{-t}\,dt, \quad s\in\mathbb{C}$$

Remark. We discuss the *gamma function* below.

Exponential Generating Functions

You may have noticed that many examples in this course involve factorials .What is the factorial function? Is there some easier way to work with it?

Let n be a nonnegative integer. The **factorial** is defined by rule

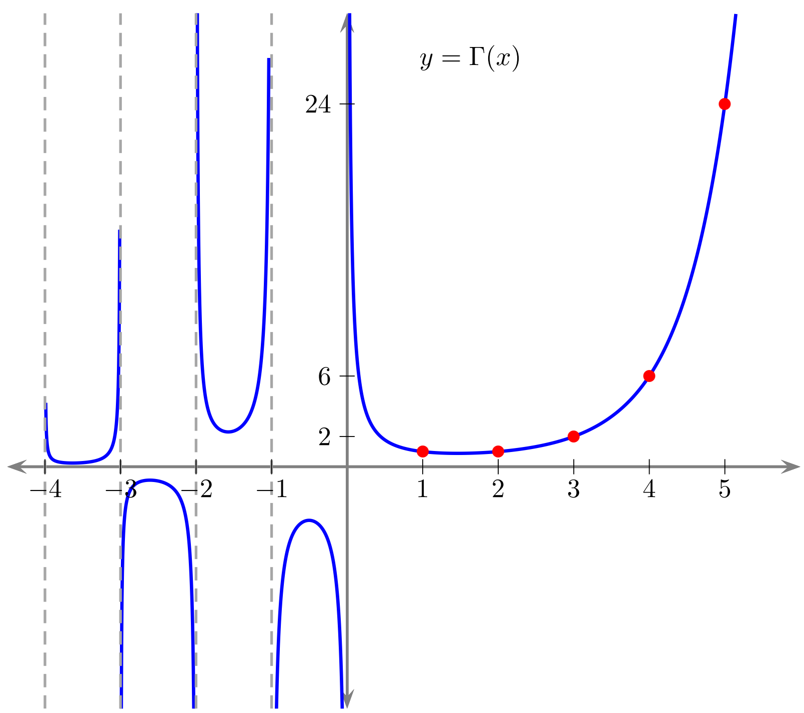
$$
\begin{aligned}
0! &= 1 \\
n! &= n(n-1)\cdots 2\cdot 1, \quad n > 0
\end{aligned}
$$

At the beginning of the 18th century, mathematicians set about looking to (continuously) extend the factorial to non-integer values. As usual it was Euler who found the solution.

Definition. The Gamma Function

$$
\Gamma(x) = \Gamma(x,0) = \int_0^\infty t^{x-1}e^{-t}\,dt \tag{12}
$$

Here the (improper) integral converges absolutely for all $x \in \mathbb{R}$ except for the non-positive integers. In fact, the Gamma function can be extended throughout the complex plane (again, except for the non-positive integers).



Observe that

$$
\begin{aligned}
\Gamma(1) &= \int_0^\infty t^0 e^{-t}\,dt \\
&= \frac{-1}{e^t}\Bigg|_0^\infty = 0-(-1) = 1
\end{aligned}
$$

and for positive integers n, integration by parts yields the recursive relation

$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty t^n e^{-t}\,dt \\ &= -t^n e^{-t}\Big|_0^\infty + n\int_0^\infty t^{n-1}e^{-t}\,dt \\ &= 0 + n\Gamma(n)\end{aligned}$$

and Euler had found his extension. That is, for each nonnegative integer n, he could now define the factorial by

$$n! = \Gamma(n+1) \tag{13}$$

The gamma function shows up in numerous formulas and important identities. For example, we have the so-called reflection formula

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

This leads to the amusing identity

$$\frac{1}{x\Gamma(x)\,\Gamma(1-x)} = \frac{\sin \pi x}{\pi x}$$

which relates the gamma function to the **sinc** function.

About the same time James Stirling discovered an *asymptotic* formula for the factorial function. Although his formula is only an approximation, these estimates do improve as $n \to \infty$ making the formula well suited for estimating the factorial for large integers. In particular, the formula can be quite useful when studying the asymptotics of a difficult sequence when an exact formula is unavailable.

Theorem. Stirling's Formula

$$n! = \Gamma(n+1) \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \tag{14}$$

Here the symbol $f(n) \sim g(n)$ means that $\lim_{n\to\infty} \dfrac{f(n)}{g(n)} = 1$.

Example 5. Evaluate $\lim_{n\to\infty} \dfrac{4^n n!n!}{(2n)!}$.

First we try it *without* Stirling's Formula. Doh!

On the other hand,

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{4^n n!n!}{(2n)!} \\
&= \frac{2\pi}{\sqrt{2\pi}} \lim_{n\to\infty} \frac{4^n}{1} \left(\frac{n}{e}\right)^n \left(\frac{n}{e}\right)^n \frac{n}{\sqrt{2n}} \left(\frac{e}{2n}\right)^{2n} \\
&= \sqrt{\pi} \lim_{n\to\infty} \frac{4^n \sqrt{n}}{1} \frac{e^{2n}}{e^n\, e^n} \frac{n^n\, n^n}{n^{2n}} \frac{1}{2^{2n}} \\
&= \sqrt{\pi} \lim_{n\to\infty} \sqrt{n} = \infty
\end{aligned}$$

As an added benefit we see that $a_n \sim \sqrt{\pi n}$ as $n \to \infty$. See Figure 1.

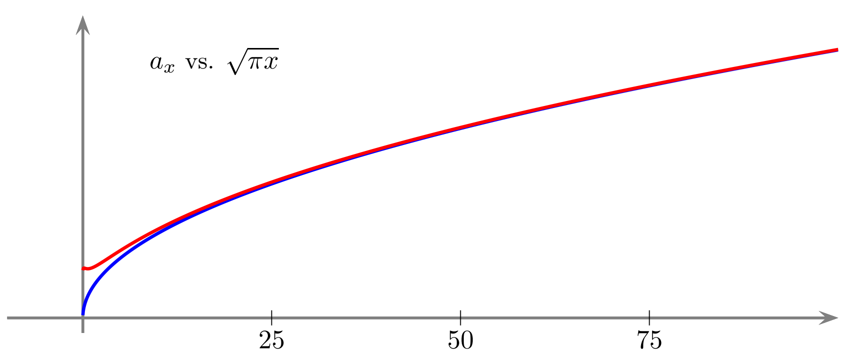


Figure 1: Sketch of the graph of $\dfrac{4^x \Gamma(x+1)\Gamma(x+1)}{\Gamma(2x+1)}$ versus $\sqrt{\pi x}$

What is arguably the most famous sequence in mathematics?