The Wilf Rules

Definition. We will use the symbol $f \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0}$ to mean that mean that f is the ordinary power series generating function for the sequence $\{a_n\}_{n\geq 0}$. That is, $f = \sum_n a_n x^n$. In a similar manner, we will use the symbols $g \xleftrightarrow{\text{egf}} \{b_n\}_{n\geq 0}$ and $h \xleftrightarrow{\text{dgf}} \{c_n\}_{n\geq 1}$ to mean that g is the exponential generating function of the sequence $\{b_n\}_{n\geq 0}$ and h is the Dirichlet series generating function of the sequence $\{c_n\}_{n\geq 1}$, respectively. Formally, we have

$$f \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0} \quad \Longrightarrow \quad f(x) = \sum_{n\geq 0} a_n x^n \tag{1}$$

$$b \xleftrightarrow{\text{egf}} \{b_n\}_{n\geq 0} \quad \Longrightarrow \quad g(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!} \tag{2}$$

$$h \xleftrightarrow{\text{dgf}} \{c_n\}_{n\geq 1} \quad \Longrightarrow \quad h(s) = \sum_{n\geq 1} \frac{c_n}{n^s} \tag{3}$$

Now suppose that $f \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0}$. What is the generating function for the sequence $\{a_{n+2}\}_{n\geq 0}$? We have

$$\begin{aligned}\sum_{n\geq 0} a_{n+2}x^n &= \frac{x^2}{x^2}\sum_{n\geq 0} a_{n+2}x^n \\ &= \frac{1}{x^2}\sum_{n\geq 0} a_{n+2}x^{n+2} \\ &= \frac{f - a_0 - a_1 x}{x^2}\end{aligned}$$

In other words,

$$f \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0} \quad \Longrightarrow \quad \frac{f - a_0 - a_1 x}{x^2} \xleftrightarrow{\text{ogf}} \{a_{n+2}\}_{n\geq 0}$$

This gives us our first rule.

Rule 1. If $f \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0}$ and k is a positive integer, then

$$\{a_{n+k}\}_{n\geq 0} \xleftrightarrow{\text{ogf}} \frac{f - a_0 - a_1 x - a_2 x^2 - \cdots a_{k-1}x^{k-1}}{x^k} \tag{4}$$

Example 1. Find a closed form for the generating function of the recursion equation below. (See additional problem 1 on Homework Set 1.)

$$a_{n+2} = -3a_{n+1} - a_n, \quad a_0 = 1, a_1 = -3 \tag{5}$$

Let $A \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0}$, then by Rule 1 we have

$$\frac{A(x) - 1 - (-3)x}{x^2} = -3\left(\frac{A(x) - 1}{x}\right) - A(x)$$

Clearing fractions we obtain

$$A(x) - 1 + 3x = -3xA(x) + 3x - x^2A(x)$$

Finally, we have

$$A(x) = \frac{1}{1 + 3x + x^2}$$

Remark. It is worthwhile to make a few observations about the last example.

1. An easy computation confirms that $A(0) = a_0 = 1$ and $A'(0) = a_1 = -3$.
2. Compare the coefficients with the denominator of $A(x)$ with the *homogeneous* form of (5).
3. Suppose that $B \xleftrightarrow{\text{ogf}} \{b_n\}_{n\geq 0}$. Can we (quickly) recover the recursion equation for the sequence? We illustrate in the next example.

Example 2. Suppose that $B \xleftrightarrow{\text{ogf}} \{b_n\}_{n\geq 0}$. Find the recursion equation for the given sequence if

$$B(x) = \frac{2 - x}{1 - 4x + x^2}$$

Observe that $B(0) = 2$ and $B'(0) = 7$. We claim that $\{b_n\}_{n\geq 0}$ satisfies the recusion

$$b_{n+2} - 4b_{n+1} + b_n = 0, \quad b_0 = 2, b_1 = 7 \tag{6}$$

Rather than working with (6) and Rule 1 to recover the generating function $B(x)$, we appeal to technology to confirm.

Compare the coefficients of the Taylor series of $B(x)$ here: `https://tinyurl.com/yas89n3g` with the first few terms generated using the recursion equation (6) here: `https://tinyurl.com/y8psbjfn`. You may need to scroll down the page to find the first few terms in the sequence. As an added bonus, the second URL also produces the closed form solution of the recursion equation (6). It is

$$b_n = \frac{\beta^{n+1} + \beta^{-n-1}}{2}, \quad \beta = 2 + \sqrt{3}$$

Moving onto the next rule, what is the effect of multiplying a given sequence by n? Once again suppose that $f \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0}$. What is the generating function for the sequence $\{na_n\}_{n\geq 0}$? We have

$$\begin{aligned}\sum_{n\geq 0} na_n x^n &= x \sum_{n\geq 0} na_n x^{n-1} = xD\left(\sum_{n\geq 0} a_n x^n\right) \\ &= xD(f(x))\end{aligned}$$

In other words,

$$f \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0} \quad \Longrightarrow \quad (xD)f \xleftrightarrow{\text{ogf}} \{na_n\}_{n\geq 0}$$

Continuing, for a positive integer k, we have

$$f \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0} \quad \Longrightarrow \quad (xD)^k f \xleftrightarrow{\text{ogf}} \{n^k a_n\}_{n\geq 0}$$

This leads to a more general form which we summarize as

Rule 2. If $f \xleftrightarrow{\text{ogf}} \{a_n\}_{n\geq 0}$ and P is a polynomial, then

$$\{P(n)a_n\}_{n\geq 0} \xleftrightarrow{\text{ogf}} P(xD)f = P(\theta)f \tag{7}$$

Note: Recall that we introduced the θ operator when we discussed the Stirling numbers.

Example 3. Find the closed form of the generating functions for the sequences given below.

a. Let $f \overset{\text{ogf}}{\longleftrightarrow} \{n^2\}_{n\geq 0}$. Then

$$\begin{aligned} f(x) &= (xD)^2 \frac{1}{1-x} \\ &= xD \frac{x}{(1-x)^2} \\ &= \frac{x(1+x)}{(1-x)^3} \end{aligned}$$

as we have seen before.

b. Let $\{f_n\}_{n\geq 0}$ be the Fibonacci numbers. Then $\{f_n\}_{n\geq 0} \overset{\text{ogf}}{\longleftrightarrow} F(x) = (1-x-x^2)^{-1}$. So by Rule 2

$$\begin{aligned} G(x) = \sum_{n\geq 0} g_n x^n &= \sum_{n\geq 0} n f_n x^n \\ &= (xD) \frac{1}{1-x-x^2} \\ &= \frac{x(1+2x)}{(1-x-x^2)^2} \end{aligned}$$

Using a CAS, it is easy to confirm that the Taylor series coefficients of $G(x)$ are

$$\begin{aligned} \{g_n\} &= \{0, 1, 4, 9, 20, \cdots\} \\ &= \{0\cdot 1, 1\cdot 1, 2\cdot 2, 3\cdot 3, 4\cdot 5, 5\cdot 8, \ldots\} \\ &= \{0f_0, 1f_1, 2f_2, 3f_3, 4f_4, 5f_5, \ldots\} \end{aligned}$$

as expected.

The next rule is simply a statement about the product of two generating functions, which we discussed in section 2.1.

Rule 3. If $f \overset{\text{ogf}}{\longleftrightarrow} \{a_n\}_{n\geq 0}$ and $g \overset{\text{ogf}}{\longleftrightarrow} \{b_n\}_{n\geq 0}$, then

$$fg \overset{\text{ogf}}{\longleftrightarrow} \left\{ \sum_{k=0}^{n} a_k b_{n-k} \right\}_{n\geq 0} \tag{8}$$

From Rule 3 we quickly general to powers of generating functions to produce
Rule 4. If $f \overset{\text{ogf}}{\longleftrightarrow} \{a_n\}_{n\geq 0}$ and k is a positive integer, then

$$f^k \overset{\text{ogf}}{\longleftrightarrow} \left\{ \sum_{n_1+n_2+\cdots n_k=n} a_{n_1}a_{n_2}\cdots a_{n_k} \right\}_{n\geq 0} \tag{9}$$

The derivation of the last rule is straightforward. See the text.
Rule 5. If $f \overset{\text{ogf}}{\longleftrightarrow} \{a_n\}_{n\geq 0}$, then

$$\frac{1}{1-x}\cdot f \overset{\text{ogf}}{\longleftrightarrow} \left\{ \sum_{k=0}^{n} a_k \right\}_{n\geq 0} \tag{10}$$

We make one additional rule that, for some reason, is missing from the Wilf list.
Rule 6. If $f \overset{\text{ogf}}{\longleftrightarrow} \{a_n\}_{n\geq 0}$ and let $a_n = 0$ for $n < 0$. Then for any positive integer k,

$$x^k f \overset{\text{ogf}}{\longleftrightarrow} \{a_{n-k}\}_{n\geq 0} \tag{11}$$

We demonstrate the utility of the rules below.

Example 4.
Define $\delta_{00} = 1$ and $\delta_{k0} = 0$ for $k > 0$. Notice that $1 \overset{\text{ogf}}{\longleftrightarrow} \{\delta_{n0}\}_{n\geq 0}$. Now by Rule 5 we have

$$\frac{1}{1-x} \cdot 1 \overset{\text{ogf}}{\longleftrightarrow} \left\{\sum_{j=0}^{n} \delta_{j0}\right\}_n = \{\delta_{00}\}_n = \{1\}_n$$

Continuing, we have

$$\frac{1}{(1-x)^2} \overset{\text{ogf}}{\longleftrightarrow} \left\{\sum_{j=0}^{n} 1\right\}_n = \{n+1\}_n$$

as we saw in class. Finally

$$\frac{1}{(1-x)^3} \overset{\text{ogf}}{\longleftrightarrow} \left\{\sum_{j=0}^{n} (j+1)\right\}_n = \left\{\frac{(n+1)(n+2)}{2}\right\}_n$$

Compare with Example 3.

Example 5. How many ways can a semester of n days be created so that there is one holiday during the first part of the semester and two holidays during the second part? Let s_n be the number of ways to do this and let k be the number of days in the first part. Clearly, we must have $1 \leq k \leq n-2$. Now for a fixed k there are $\binom{k}{1} = k$ ways to choose a holiday during the first part of the semester and $\binom{n-k}{2}$ to choose the pair of holidays during the second part. So by the product rule, there are $k\binom{n-k}{2}$ ways to create the semester. Summing over all values of k yields

$$s_n = \sum_{k=1}^{n-2} k \binom{n-k}{2} \tag{12}$$

ways to create a semester of n days under the given conditions.

Equation (12) appears to be a satisfactory answer, but it does not give s_n in a closed form, if one even exists. The appearance of a convolution in (12) strongly suggests that it might helpful to look at products of generating functions. So let $a_n = n$ and $b_n = \binom{n}{2}$ and let $S(x) = \sum_{n\geq 3} s_n x^n$. Also, let $A(x) = \sum_n a_n x^n$ and $B(x) = \sum_n b_n x^n$. We leave it as an exercise to show that $S(x) = A(x)B(x)$.

Now recall the binomial formulas discussed earlier in the semester.

$$\sum_n \binom{n}{k} = \frac{x^k}{(1-x)^{k+1}} \tag{13}$$

It follows that

$$A(x) = \sum_n a_n x^n = \sum_n \binom{n}{1} x^n = \frac{x}{(1-x)^2} \tag{14}$$

$$B(x) = \sum_n b_n x^n = \sum_n \binom{n}{2} x^n = \frac{x^2}{(1-x)^3} \tag{15}$$

We have

$$S(x) = A(x)B(x) = \frac{x^3}{(1-x)^5}$$

It follows that

$$\begin{aligned} s_n &= [x^n]\frac{x^3}{(1-x)^5} = [x^n]\frac{1}{x}\,\frac{x^4}{(1-x)^5} \\ &= [x^{n+1}]\sum_n \binom{n}{4} x^n = \binom{n+1}{4} \end{aligned}$$

In other words,

$$\sum_{k=1}^{n-2} k\binom{n-k}{2} = \binom{n+1}{4}$$

and we have our closed form.

We finish this section with a beautiful theorem about the general nature of the recursion equation for *rational* generating functions.

Theorem 6. Let p and q be polynomials which are relatively prime and with $0 < k = \deg p < \deg q = m$. Suppose also that

$$q(x) = \sum_{j=0}^{m} q_j x^j, \quad \text{with } q_0 \neq 0$$

Finally, let

$$f(x) = \frac{p(x)}{q(x)} \overset{\text{ogf}}{\longleftrightarrow} \{f_n\}_{n\geq 0}$$

Then for all $n \geq 0$, the sequence of coefficients f_n satisfies the recursion

$$q_0 f_{n+m} + q_1 f_{n+m-1} + \cdots + q_m f_n = 0 \tag{16}$$

with initial conditions given by

$$f_j = \frac{f^{(j)}(0)}{j!}, \quad 0 \leq j \leq m-1 \tag{17}$$

Here $f^{(j)}(0)$ is the jth derivative of f evaluated at zero.

Proof: Observe that for $n \geq 0$, we have that $[x^{n+m}]p(x) = 0$ since p is a polynomial with $\deg p < m$. It follows that

$$\begin{aligned} 0 = [x^{n+m}]p(x) = [x^{n+m}]f(x)q(x) &= [x^{n+m}]f(x)\sum_{j=0}^{m} q_j x^j \\ &= \sum_{j=0}^{m} q_j [x^{n+m}]x^j f(x) = \sum_{j=0}^{m} q_j [x^{n+m-j}]f(x) \\ &= \sum_{j=0}^{m} q_j f_{n+m-j} \end{aligned}$$

which is (16). Notice that (17) follows by Taylor's Theorem. $\square$

Example 7. Let $G \overset{\text{ogf}}{\longleftrightarrow} \{g_n\}_n$ where

$$G(x) = \frac{1}{1 - ax - bx^2 - cx^3}, \qquad c \neq 0$$

Then by Theorem 6, the sequence of coefficients g_n satisfies the recursion

$$g_{n+3} = ag_{n+2} + bg_{n+1} + cg_n, \quad n \geq 0 \tag{18}$$

with initial conditions

$$\begin{aligned} g_0 &= G(0) = 1 \\ g_1 &= G'(0) = a \\ g_2 &= \frac{G''(0)}{2!} = \frac{2(a^2 + b)}{2} \end{aligned}$$