1. (10 points) Let $F(x) \overset{\text{ogf}}{\longleftrightarrow} \{f_n\}_{n\geq 0}$ and $G(x) \overset{\text{ogf}}{\longleftrightarrow} \{g_n\}_{n\geq 0}$. Suppose that $g_n = \sum_k \binom{n}{k} (-1)^{n-k} f_k$. Write $G(x)$ in terms of $F(x)$.

Solution:

By binomial inversion, we know that $f_n = \sum_k \binom{n}{k} g_k$, and we showed that in this case,

$$F(x) = \frac{1}{1-x} G\left(\frac{x}{1-x}\right) \qquad \text{(Why?)} \tag{1}$$

It turns out that

$$G(x) = \frac{1}{1+x} F\left(\frac{x}{1+x}\right) \tag{2}$$

To see this, we could mimic the proof that we used to derive (1). Instead, we let $y = x/(1-x)$ in (1). It follows that

$$x = \frac{y}{1+y}$$

and (1) implies

$$\begin{aligned} G(y) &= \left(1 - \frac{y}{1+y}\right) F\left(\frac{y}{1+y}\right) \\ &= \left(\frac{1+y}{1+y} - \frac{y}{1+y}\right) F\left(\frac{y}{1+y}\right) \\ &= \frac{1}{1+y} F\left(\frac{y}{1+y}\right) \end{aligned}$$

and (2) is established.

2. (10 points) A coach wishes to break up her n-member team into 3 practice squads. Each player on squad A will wear either a red or a blue jersey, those on squad B will wear yellow jerseys numbered from 1 to |squad B|, and squad C players will wear black jerseys and choose a squad captain. Let $t_0 = 0$ and for $n > 0$, let t_n count the number of ways that she can do this. Find the closed form of the exponential generating function $\sum_n t_n\, x^n/n!$. *Note:* This means that the squad B team is ordered and that squad C must have at least one player.

   Hint: The first few terms in this sequence are $0, 1, 8, 51, 312, \ldots$

   Solution:

   Let $i, j,$ and k be the number of players resp. on squad A, squad B, and squad C. Then

$$t_n = \sum_{i+j+k=n} \frac{n!}{i!j!k!} 2^i\, j!\, k \tag{3}$$

   So by the Wilf rules, we must have

$$\begin{aligned} T(x) = \sum_n t_n \frac{x^n}{n!} &= \sum_n 2^n \frac{x^n}{n!} \sum_n n! \frac{x^n}{n!} \sum_n n \frac{x^n}{n!} \\ &= e^{2x} \frac{1}{1-x}\, x e^x = \frac{xe^{3x}}{1-x} \end{aligned}$$

   Although it wasn't requested, we can see that

$$t_n = n![x^n] \frac{xe^{3x}}{1-x} = n![x^{n-1}] \frac{1}{1-x} e^{3x} = n! \sum_{k=0}^{n-1} \frac{3^k}{k!}$$

   So, for example,

$$t_3 = 3! \left(1 + \frac{3}{1} + \frac{3^2}{2!}\right) = 6(1 + 3 + 9/2) = 51, \text{ as expected}$$

   It is also worthwhile to list the 8 possibilities with two players, say 1 and 2. For example, $1^r|0|2$ indicates that player 1 is given a red jersey on squad A and player 2 would be the captain of squad C. Here are the other 7 lineups.

$$1^b|0|2,\ 2^r|0|1,\ 2^b|0|1$$
$$0|1|2,\ 0|2|1$$
$$0|0|1^c2,\ 0|0|12^c$$

   Hopefully, the notation is self-explanatory and it should be clear that these are the only cases.

   Remark. (a) Notice that the right-hand side of (3) is zero whenever $k = 0$. That is, for any configuration that assigns zero players to squad C, the summand is 0, as expected.

   (b) It is a worthwhile exercise to list the 51 possibilities for 3 players.