1. (8 points) Let $f_n = [x^n]\dfrac{7x}{(2-3x)(1+2x)}$. Find a closed formula for f_n.

Solution:

A routine partial fraction decomposition produces

$$\frac{7x}{(2-3x)(1+2x)} = \frac{2}{2-3x} - \frac{1}{1+2x}$$

Thus

$$\begin{aligned} f_n &= [x^n]\frac{2}{2-3x} - [x^n]\frac{1}{1+2x} \\ &= [x^n]\frac{1}{1-3x/2} - [x^n]\frac{1}{1+2x} \\ &= \left(\frac{3}{2}\right)^n - (-2)^n \end{aligned}$$

2. Let $\{f_n\}_{n\geq 0}$ denote the Fibonacci numbers and consider the following identity.

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1,\ n \geq 0 \tag{1}$$

(a) (5 points) Use generating functions to prove (1).

Solution:

Let $F(x) = \sum_{n\geq 0} f_n x^n = (1-x-x^2)^{-1}$. For the left-hand side of (1), the Wilf rules yield

$$\frac{F(x)}{1-x} = \frac{1}{1-x}\,\frac{1}{1-x-x^2} \tag{2}$$

Applying these rules to the right-hand side produces

$$\begin{aligned}\frac{1}{x^2}(F(x)-1-x) - \frac{1}{1-x} &= \frac{1}{x^2}\left(\frac{1}{1-x-x^2} - 1 - x\right) - \frac{1}{1-x}\\ &= \frac{1}{x^2}\,\frac{2x^2+x^3}{1-x-x^2} - \frac{1}{1-x}\\ &= \frac{2+x}{1-x-x^2} - \frac{1}{1-x}\\ &= \frac{1}{1-x}\,\frac{1}{1-x-x^2}\end{aligned}$$

in agreement with (2).

(b) (5 points) Find a combinatorial proof of (1).

Solution:

The right-hand side counts the number of ways to cover B_{n+2} (see Exercise 11/03 - #3) using at least one domino.

For the left-hand side, we condition on the starting position of the last domino. There is only $1 = f_0$ way to cover the board if the last domino starts a tile 1. In general, if the last domino starts at tile $k+1$, then there are f_k ways to cover the first k tiles (and only one way to cover the remaining tiles). These are clearly disjoint, so summing over k from 0 to n yields the left-hand side of (1).

(c) (2 points) Let $g_n = \sum_{k=0}^{n} f_k$. What recursive equation does g_n satisfy?

Solution:

From (1) we have

$$\begin{aligned}g_{n+2} &= f_{n+4} - 1\\ &= f_{n+3} + f_{n+2} - 1\\ &= (g_{n+1}+1) + (g_n + 1) - 1\\ &= g_{n+1} + g_n + 1, \quad g_0 = 1, g_1 = 2\end{aligned}$$