1. (12 points) Consider the recursion below and answer the questions that follow.

$$a_{n+2} = 3a_{n+1} - a_n, \quad a_0 = 1,\ a_1 = 4 \tag{1}$$

(a) *Carefully* find the next 4 terms in this sequence.

Solution:

$a_2 = 3(4) - 1 = 11$, etc., so the next 4 terms are 11, 29, 76, 199. Here are the first 11 terms.

(b) Find the closed form of the generating function for this sequence, that is, find the closed form of $A(x) = \sum_{n\geq 0} a_n x^n$.

Solution:

We can multiply the recurrence in (1) by x^{n+2} and sum over n to produce

$$\sum_{n\geq 0} a_{n+2}x^{n+2} = 3\sum_{n\geq 0} a_{n+1}x^{n+2} - \sum_{n\geq 0} a_n x^{n+2}$$

This is equivalent to

$$A(x) - a_0 - a_1 x = 3x(A(x) - a_0) - x^2 A(x)$$

Note: One can (and probably should) use the Wilf rules to produce the same equation.

Now apply the initial conditions and rearrange to obtain

$$A(x)(1 - 3x + x^2) = 1 + 4x - 3x$$

Thus

$$A(x) = \frac{1 + x}{1 - 3x + x^2}$$

Compare the coefficients in the Taylor Series expansion of $A(x)$ with the terms listed in part (a).

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Instructions. Grading is based on method. Show all work.

Submit solutions at the beginning of class on Monday.

2. (8 points) Let $\{b_n\}_{n\geq 0}$ be the sequence defined by the recursion below.

$$b_{n+3} = 3b_{n+2} - b_{n+1} + 4b_n, \quad b_0 = 1,\ b_1 = 5,\ b_2 = 12 \tag{2}$$

According to Stanley's Theorem, the closed form of the ordinary generating function whose sequence of coefficients satisfies (2) must be of the form

$$B(x) = \sum_{n\geq 0} b_n x^n = \frac{q(x)}{1 - 3x + x^2 - 4x^3} \tag{3}$$

where $q(x)$ is a nonzero polynomial of degree strictly less than 3.

Find $q(x)$ and $B(x)$.