1. (10 points) A local restaurant sent out 38 tablecloths to be laundered. Each tablecloth had zero or more stains from coffee (C), vinegar (V), grease (G), and pizza sauce (S) as indicated in the table below.

Stain	Count	Stain	Count
V	21	G	19
S	20	C	12
VC	6	VG	12
SG	10	CS	5
CG	7	VS	11
VCS	3	VCG	4
VGS	6	CSG	3

If two of the tablecloths had all 4 stains, answer the question below.

(a) How many tablecloths had no stains?

Solution:

Lets first construct $N(x)$. According to the table,

$$N(x) = 38 + 72x + 51x^2 + 16x^3 + 2x^4$$

It follows that

$$\begin{aligned} E(x) &= N(x-1) \\ &= 3 + 10x + 15x^2 + 8x^3 + 2x^4 \end{aligned}$$

Now it's easy to see that there are exactly 3 shirts with no stains.

(b) How many tablecloths had exactly one stain?

Solution:

From part (a), we see that the number of shirts with exactly one stain is $e_1 = [x^1]E(x) = 10$.

2. (10 points) Find the number of nonnegative integer solutions to the equation below subject to the restrictions that follow.

$$x_1 + x_2 + x_3 = 30, \tag{1}$$

where $0 \leq x_1 \leq 9$, $3 \leq x_2 \leq 14$, and $0 \leq x_3 \leq 17$. EXPRESS YOUR ANSWER AS AN INTEGER.

Recall that the number of *nonnegative* integer solutions to (1) (without these restrictions) is given by the multichoose coefficient $\left(\!\binom{3}{30}\!\right)$.

Solution:

Throughout this proof, solutions to (1) will always mean nonnegative integer solutions.

First let $y_2 = x_2 - 3$. Then we may rewrite (1) as

$$x_1 + y_2 + 3 + x_3 = 30 \quad \text{or} \quad x_1 + y_2 + x_3 = 27 \tag{2}$$

with x_1 and x_3 restrictions as above and $0 \leq y_2 \leq 11$.

So let p_1 be the condition that $x_1 \geq 10$ and let $N(p_1)$ count the number of nonnegative solutions to the equation $x_1 + y_2 + x_3 = 27$ with the added restriction to x_1. It is easy to see that this is equivalent to the number of nonnegative solutions to $x_1' + 10 + y_2 + x_3 = 27$ or $x_1' + y_2 + x_3 = 17$ so that $N(p_1) = \left(\!\binom{3}{17}\!\right)$. In a similar manner, let p_2 be the condition that $y_2 \geq 12$ and p_3 be the condition that $x_3 \geq 18$. Then $N(p_2) = \left(\!\binom{3}{15}\!\right)$ and $N(p_3) = \left(\!\binom{3}{9}\!\right)$. Easy calculations show that $N(p_1p_2) = \left(\!\binom{3}{5}\!\right)$, $N(p_1p_3) = 0$, and $N(p_2p_3) = 0$. Finally, $N(p_1p_2p_3) = 0$.

Then the number of solutions the (2) under the initial restrictions is given by

$$\begin{aligned} N_0 &= \left(\!\binom{3}{27}\!\right) - \left(\left(\!\binom{3}{17}\!\right) + \left(\!\binom{3}{15}\!\right) + \left(\!\binom{3}{9}\!\right)\right) + \left(\left(\!\binom{3}{5}\!\right) + 0 + 0\right) - 0 \\ &= \binom{29}{27} - \left(\binom{19}{17} + \binom{17}{15} + \binom{11}{9}\right) + \binom{7}{5} \\ &= 65 \end{aligned}$$