1. (4 points) Let $T_n \subseteq \mathfrak{S}_n$ denote the set of permutations whose cube is the identity. For example, $\pi = (124)(3) \in T_4$ since $\pi^3 = \text{id}$. *Note:* Such permutations can only be comprised of 1-cycles and 3-cycles.

   It is easy to see that $T_1 = \{(1)\}, T_2 = \{(1)(2)\}$ and $T_3 = \{(1)(2)(3), (123), (132)\}$. Use cycle notation to list the other 8 elements of T_4.

   Solution:

   The remaining permutations are

   $$(123)(4), (134)(2), (1)(234), (132)(4), (143)(2), (1)(243), (142)(3), (1)(2)(3)(4)$$

2. (4 points) Let $\sigma \in \mathfrak{S}_7$ with inversion table $\sigma_I = 3002110$.

   (a) Rewrite σ using one-line notation.

   Solution:

   $\sigma = (2\ 3\ 7\ 1\ 5\ 4\ 6)$

   (b) Rewrite σ using cycle notation.

   Solution:

   $\sigma = (123764)(5)$

3. (5 points) Let $\pi = (\pi_1\ \pi_2\ \cdots\ \pi_n) \in \mathfrak{S}_n$ be a permutation and let $E(\pi)$ be the set of its inversions. Prove that $E(\pi)$ is transitive. That is, prove that if (a, b) and (b, c) are in $E(\pi)$, then $(a, c) \in E(\pi)$.

   Solution:

   This is rather straight-forward. If (a, b) and (b, c) are in $E(\pi)$, then $a > b$ and $b > c$, hence $a > c$. Now if π is written in the usual one-line notation, a lies to the left of b and b lies to the left of c. In other words, a lies to the left of c and so $(a, c) \in E(\pi)$.

4. (7 points) In how many ways can the 12 letters $\{a^5, b^4, c^3\}$ be arranged so that there are not 5 consecutive a's, nor 4 consecutive b's, nor 3 consecutive c's. For example, *abcccabababa* and *caaaaacbbbbc* are forbidden strings, but the string *aaaabbbccabc* is legal. *Note:* There are several ways to compute this. I used the principle of inclusion and exclusion.

Solution:

Following the hint, let $p_x,\ x \in \{a, b, c\}$ correspond to the property that all identical letters are consecutive. Now let N count the number of distinguishable permutations of the 12 letters, then
$N = 12!/5!/4!/3! = 27720$.

What is $N(p_a)$? To answer this, we treat the a's as a single block, call it A. So how many ways are there to arrange the 8 letters $\{A, b^4, c^3\}$. Once again, we appeal to our formula for distinguishable permutations to conclude that

$$N(p_a) = \frac{8!}{1!4!3!} = 280$$

Similarly

$$N(p_b) = \frac{9!}{5!1!3!} = 504 \text{ and } N(p_c) = \frac{10!}{5!4!1!} = 1260$$

Using similar arguments, it follows that

$$N(p_a p_b) = \frac{5!}{3!} = 20$$
$$N(p_a p_c) = \frac{6!}{4!} = 30$$
$$N(p_b p_c) = \frac{7!}{5!} = 42$$
$$N(p_a p_b p_c) = 3! = 6$$

It follows by PIE that the number of permissable strings is

$$e_0 = 27720 - (280 - 504 - 1260) + (20 + 30 + 42) - 6 = 25762$$