1. (10 points) For $n \geq 3$, show that

$$\begin{bmatrix} n \\ n-2 \end{bmatrix} = \frac{3n-1}{4}\binom{n}{3} \tag{1}$$

Hint: If $\pi \in \left[\begin{smallmatrix} [n] \\ n-2 \end{smallmatrix}\right]$ then π must contain zero or more singletons and exactly two transpositions (doubletons) or it must contain zero or more singletons and exactly one 3-cycle. Here is an example of each type from $\left[\begin{smallmatrix} [8] \\ 6 \end{smallmatrix}\right]$.

$$(1)(2)(37)(4)(56)(8) \quad \text{and} \quad (1)(2)(356)(4)(7)(8)$$

Solution:

We give two proofs.

Comb: Let $\pi \in \binom{[n]}{n-2}$. Following the hint: For the first type, there are $\binom{n}{2}$ ways to choose the first transposition and $\binom{n-2}{2}$ to choose the second. Since the order of the transpositions doesn't matter, there are $\binom{n}{2}\binom{n-2}{2}/2$ ways to choose permutations of this type.

It is easy to see that there are $2\binom{n}{3}$ ways to choose permutations of the second type. Notice that the sum rule applies, so that

$$\begin{aligned} \begin{bmatrix} n \\ n-2 \end{bmatrix} &= \frac{1}{2}\binom{n}{2}\binom{n-2}{2} + 2\binom{n}{3} \\ &= \frac{1}{2}\frac{n!}{2!(n-2)!}\frac{(n-2)!}{2!(n-4)!} + 2\binom{n}{3} \\ &= 3\binom{n}{4} + 2\binom{n}{3} \\ &= \frac{3n-1}{4}\binom{n}{3} \end{aligned}$$

Algebraic: Let a_j count the number of cycles of length j, as described in Theorem 6.9 from the text.

For permutations of the first type: $a_1 = n-4, a_2 = 2$, then by Theorem 6.9, the number of such permutations is $\frac{n!}{(n-4)!2!1^{n-4}2^2}$

For permutations of the second type: $a_1 = n-3, a_3 = 1$, then by Theorem 6.9, the number of such permutations is $\frac{n!}{(n-3)!1^{n-3}3^1}$

It now follows by the sum rule that

$$\begin{aligned} \begin{bmatrix} n \\ n-2 \end{bmatrix} &= \frac{n!}{(n-4)!8} + \frac{n!}{3(n-3)!} \\ &= 3\binom{n}{4} + 2\binom{n}{3} \end{aligned}$$

as we saw above.

3. (10 points) For $n \geq 0$, show that

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} 2^k = (n+1)! \tag{2}$$

Solution:

In class we showed (see Lemma 6.13 from the text) that

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k = x^{\overline{n}}$$

It follows that

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} 2^k = 2^{\overline{n}} = 2(2+1)(2+2)\cdots(2+n-1) = (n+1)!$$

There is more that we can learn from this problem. Recall the following result from Friday.

$$\begin{bmatrix} n+1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix} = \sum_{k=1}^{n} k \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{1} \tag{3}$$

We proved the middle identity in class (the outside equalities are trivial).

Returning to the problem above and rewriting the left-hand side of (2) produces

$$\begin{aligned} (n+1)! &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix} 2^k \\ &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \sum_m \binom{k}{m} \\ &= \sum_m \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{m} \end{aligned}$$

If we now compare the inside sum of the last line to the right-hand side of (3), we might conclude that

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{m} = \begin{bmatrix} n+1 \\ m+1 \end{bmatrix} \tag{4}$$

And this identity turns out to be true. The proof is nearly identical to the one we gave for (3) in class. The only difference is that we <u>identify</u> m cycles in $\mathfrak{S}_n$ instead of just one. Of course, (3) would then be a special case of (4).