1. (3 points) How many terms are in the sum below? Justify your response.

$$\sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

For example, $\sum_{n_1+n_2=3} \binom{3}{n_1,n_2} x_1^{n_1} x_2^{n_2} = x_1^3 + 3x_1^2 x_2 + 3x_1 x_2^2 + x_2^3$ is a sum of 4 terms.

Solution:

There are as many terms as there are nonnegative integer solutions to the equation $n_1 + n_2 + \cdots + n_k = n$. But this is $\left(\!\binom{k}{n}\!\right)$, as we saw in chapter 3.

2. (7 points) Give a combinatorial proof of the identity below.

$$4^k = \sum_{j=0}^{k} 3^j \binom{k}{j}$$

Solution:

The left-hand side counts the number of k-words on the alphabet $\mathcal{A} = \{a, b, c, d\}$.

For the right-hand side, there are $\binom{k}{k-j} = \binom{k}{j}$ ways to position $k - j$ occurrences of the letter a. Now the remaining positions may be filled with letters from the alphabet $\{b, c, d\}$ in 3^j ways. So by the product rule, there are $\binom{k}{j}3^j$ ways to create a k-word with exactly $k - j$ occurrences of a. Now we sum over j as j ranges from 0 to k since these collections are disjoint. The result follows.

3. (10 points) Consider the 9-letter "word", HTHHTTHHH and answer the questions below.

   (a) How many <u>different</u> arrangements are possible?

   Solution:

   This is just distinguishable permutations. It follows that there are

   $$\frac{9!}{6!3!} = 84 \text{ such arrangements.}$$

   (b) How many different arrangements are possible if consecutive T's are forbidden? JUSTIFY YOUR RESPONSE.

   Solution:

   How many ways can the T's be distributed across the spaces below (with no more than one T per space)?

   $$\_\,H\,\_\,H\,\_\,H\,\_\,H\,\_\,H\,\_\,H\,\_$$

   This is just $\binom{7}{3} = 35$. To see this, consider the following allowable arrangement (word): HHHTHHTHT. One way to specify this arrangement would be number the open slots as suggested above.

   $$\underline{\;1\;}\,H\,\underline{\;2\;}\,H\,\underline{\;3\;}\,H\,\underline{\;4\;}\,H\,\underline{\;5\;}\,H\,\underline{\;6\;}\,H\,\underline{\;7\;}$$

   Now the word HHHTHHTHT would correspond to the subset $\{4, 6, 7\} \subset [7]$. So we have reduced the question to finding all subsets of [7] of size 3. That is, $\binom{7}{3}$.