1. (6 points) For $0 < m < k < n$, use a combinatorial argument to prove the identity below. No credit for any other method.

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

Hint: If $m = 1$ this is the extraction/absorbtion identity. A similar proof will work here.

Solution:

Following the hint, from a group of n students, in how many ways can we choose a committee of k students and a subcommittee (from the committee) of m students?

LHS - There are $\binom{n}{k}$ ways to first choose the committee and $\binom{k}{m}$ ways to choose the subcommittee. So by the product rule, there are $\binom{n}{k}\binom{k}{m}$ ways to form a committee in this way.

RHS - There are $\binom{n}{m}$ ways to first choose the subcommittee. Now we must choose that remaining $k - m$ members of the committee from the remaining $n - m$ students and there are $\binom{n-m}{k-m}$ to do this. Once again, we appeal to the product rule.

2. (6 points) A collection of fruit contains exactly 10 items – an assortment of apples, oranges, and pears. For example, one possible collection has 3 apples, 7 oranges, and 0 pears. How many distinct collections are possible?

Solution:

This is a routine stars and bars argument and we have 10 items (stars) and 3 categories, so we need 2 dividers (bars). It follows that there are $\binom{12}{2} = 66$ distinct collections.

What happens if we add some additional restrictions. For example, how many collections are possible if we must have at least one of each fruit? Or if we only have 8 pears available?

3. (8 points) Use the identity $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ to prove the identity below.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{1}$$

NO CREDIT FOR ANY OTHER METHOD.

Solution:

If $y \neq 0$, then

$$\begin{aligned}(x+y)^n = y^n(1+x/y)^n &= y^n \sum_{k=0}^{n} \binom{n}{k} (x/y)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k \frac{y^n}{y^k}\end{aligned}$$

which is (1). If $y = 0$, the identity states that

$$(x+0)^n = \binom{n}{n} x^n$$

as expected.