

1. (10 points) Two positive integers are called *relatively prime* if their greatest common divisor is 1. For example, 12 and 25 are relatively prime but 18 and 21 are not. Now select $n + 1$ different integers from the set $[2n] = \{1, 2, 3, \dots, 2n\}$. Show that there are always two (at least) among the selection that are relatively prime.

Solution:

Notice that if $k \in \mathbb{P}$, then $\gcd(k, k + 1) = 1$, i.e., consecutive integers are relatively prime.

Now create n boxes $B_1, B_2, \dots, B_n$ and assign integers $1, 2 \in B_1, 3, 4 \in B_2, \dots, 2n - 1, 2n \in B_n$. Now if $n + 1$ balls are distributed among the n boxes (indicating that either of the integers were chosen), then by the PHP there is at least one box with two balls. In other words, consecutive integers were chosen and we are done.

2. (10 points) Prove that there exists a positive integer n so that $44^n - 1$ is divisible by 17.

Solution:

Let $f(n)$ be the remainder when $44^n - 1$ is divided by 17.

Then by the PHP, there are integers $1 \leq p < q \leq 18$ such that $f(q) = f(p)$ since there are only 17 possible remainders. It follows that

$$44^p - 1 = M_1 \cdot 17 + f(p) \quad \text{and} \quad 44^q - 1 = M_2 \cdot 17 + f(q)$$

for positive integers M_1 and M_2 . Thus

$$\begin{aligned} 44^q - 44^p &= 44^p(44^{q-p} - 1) \\ &= 17(M_2 - M_1) \end{aligned}$$

Thus 17 divides $44^p(44^{q-p} - 1)$. Clearly 17 does not divide 44^p for any $p \in \mathbb{N}$ since 17 and 44 are relatively prime. It follows that $17 \mid 44^{q-p} - 1$, as desired.